DIVISION SEMINEAR-RINGS

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We show that any finite division seminear-ring is uniquely determined by the Zappa-Szép product of two multiplicative subgroups, and classify all seminear-fields in three out of four categories.

1. Introduction

In [2] the authors investigated semifields in which the addition and multiplication are both commutative. In [3] the first author extended most of the work in [2] to the non-commutative case: this paper outlines the more significant results in [3].

2. Seminear-rings

We say that \((S, +, \cdot)\) is a right seminear-ring if \(S\) is a set with two binary operations + and \(\cdot\) such that \((S, +)\) and \((S, \cdot)\) are semigroups and the right distributive law holds: \((x + y)z = xz + yz\) for all \(x, y, z \in S\). A left seminear-ring is similarly defined, and if \(S\) is both a left and a right seminear-ring then it is a semiring. An important example of a right seminear-ring is obtained by starting with an arbitrary semigroup \((S, +)\) and letting \(M(S)\) denote the set of all maps from \(S\) into itself; if + and \(\cdot\) are defined on \(M(S)\) as pointwise addition and composition respectively, then \((M(S), +, \cdot)\) is a right seminear-ring which is not left distributive provided \(|S| > 1\).

In what follows, the word ‘seminear-ring’ will mean a ‘right seminear-ring’. A division seminear-ring is a seminear-ring \((D, +, \cdot)\) in which \((D, \cdot)\) is a group. The set \(\mathbb{R}^+\) of positive real numbers with the usual addition and multiplication is a division seminear-ring in which the left distributive

Received February 11, 1993.
law holds: that is, it is a division semiring. To obtain a family of right (but not left) division seminear-rings, we need the following notion.

If \((G, \cdot)\) is a group and \(H, K\) are subgroups of \(G\), we say \(G\) is a Zappa-Szép (ZS) product of \(H\) and \(K\), written \(G = H \ast K\), if \(G = HK\) and \(H \cap K = 1\). Note that any direct product is a ZS-product but not conversely. For example, if \(G = S_3\) and \(H = \langle (1,2) \rangle, K = A_3\) then \(G = H \ast K\) but \(S_3\) is not a direct product of any of its subgroups. The proof of the following result is straight-forward and so is omitted.

**Lemma 1.** If \(G = H \ast K\) for some subgroups \(H, K\) of a group \(G\), then \(G = K \ast H\) and

(a) for each \(x \in G\) there exist unique \(h_1, h_2 \in H\) and \(k_1, k_2 \in K\) such that \(x = h_1k_1 = k_2h_2\), and

(b) for each \(h \in H, k \in K\) there exist unique \(h' \in H\) and \(k' \in K\) such that \(h'k = k'h\).

The next result provides a way of constructing division seminear-rings which are not division semirings.

**Theorem 1.** If \(G = H \ast K\) for some subgroup \(H, K\) of group \(G\), then there exists a unique binary operation \(+\) on \(G\) such that \((G, +, \cdot)\) is a division seminear-ring in which \((G, +)\) is a rectangular band containing \(H\) and \(K\) as left and right zero subsemigroups respectively and \(G = H + K\).

**Proof.** Let \(x_1, x_2 \in G\). By Lemma 1, we can write \(x_1 = k_1h_1, x_2 = h_2k_2\) and \(h_1'k_2 = k_2'h_1\) for suitable unique elements of \(H\) and \(K\). In this case, we define \(x_1 + x_2\) to be \(h_1'k_2\).

Suppose \(x \in G\) and \(x = kh_1 = hk_1\). Then, by uniqueness and the definitions, \(h_1 = 1 \cdot h_1, k_1 = 1 \cdot k_1\) imply that \(h_1 + k_1 = hk_1 = x\). Moreover, if \(x = h + k = h'k = k'h\) then \(h'k = hk_1\) and uniqueness implies \(k = k_1\); similarly, \(h = h_1\) and we have shown that for each \(x \in G\), there are unique \(h \in H, k \in K\) such that \(x = h + k\). In addition, if \(hk_1 = kh_1\) and \(h'k_2 = k'h_2\) then \((h_1 + k_1) + (h_2 + k_2) = kh_1 + k'h_2\). Hence, if \(h''k_2 = k''h_1\) then \((h_1 + k_1) + (h_2 + k_2) = h_1 + k_2\). Now, it is well-known that \(H \times K\) under the operation:

\[(h_1, k_1) \otimes (h_2, k_2) = (h_1, k_2)\]

is a rectangular band [1]. And, from the foregoing remarks, \(f : G \rightarrow H \times K, h + k \rightarrow (h, k)\), is an isomorphism from \((G, +)\) onto \((H \times K, \otimes)\). Thus, \((G, +)\) is a rectangular band in which \(H\) is a left zero semigroup.
For, if \( h \in H \) then \( h \cdot 1 = 1 \cdot h \) implies that \( h + 1 = h \cdot 1 = h \) and so \( H \) is isomorphic under \( f \) to \( H \times 1 \), a left zero subsemigroup of \((H \times K, \otimes)\).

To show that \((G, +, \cdot)\) is right distributive, let \( x, y, z \in G \) and write

\[
xz = h_1 + k_1 = h'_1 h_1
\]

\[
yz = h_2 + k_2 = h'_2 h_2.
\]

Then, if \( x = h_3 k_3 = k_4 h_4 \) and \( y = h_5 k_5 = k_6 h_6 \), we have \( z = x^{-1} k'_1 h_1 = y^{-1} h'_2 k_2 \) and so

\[
k'_4 k'_1 h_1 = h_4 k'_5 h_5 h'_2 k_2 = k_7 h_7 \cdot h_5 h'_2 k_2
\]

for some \( h_7, k_7 \) in \( G \). Since \( h_7 h_5 h'_2 k_2 \in H \) and \( k_7 h_5 h'_2 k_1 \in K \), we conclude that \( xz + yz = h_1 + k_2 \) and this equals \( h_7 h_5^{-1} h'_2 k_2 \). But, \( x = h_4 + k_3, y = h_6 + k_5 \) and so

\[
(x + y)z = [(h_4 + k_3) + (h_6 + k_5)]y^{-1} h'_2 k_2
\]

\[
= (h_4 + k_5) (h_5 k_5)^{-1} h'_2 k_2 = h_7 h_5^{-1} h'_2 k_2
\]

\[
xz + yz
\]

since \( h_4 + k_5 = h'_4 k_5 = k'_5 h_4 \) implies that \( h'_4 = h_7 \). Finally, to show \(+\) is unique, suppose \( \oplus \) is another operation for which \((G, \oplus, \cdot)\) has the same properties as \((G, +, \cdot)\). Now, under the stated conditions, \( k + h = (1+k) + (h+1) = 1 + (k+h) + 1 = 1 \). Thus, if \( h \oplus k = h_1 k_1 = k_2 h_2 = h_2 + k_1 \). Then

\[
1 = 1 \oplus kh^{-1} \oplus 1 = (h \oplus k)h^{-1} \oplus 1 = k_2 h_2 h^{-1} \oplus 1
\]

and so \( 1 = k_2 \oplus hh_2^{-1} = (k_2 h_2 h^{-1} \oplus 1)hh_2^{-1} = hh_2^{-1} \). Hence, \( h = h_2 \) and similarly \( k = k_1 \). So, if \( x, y \in G \) satisfy \( x = h_1 \oplus k_1 = h_1 + k_1 \) and \( y = h_2 \oplus k_2 = h_2 + k_2 \) then \( x \oplus y = h_1 \oplus k_2 = h_1 + k_2 = x + y \), as required.

With the same notation as in Theorem 1, it can be shown that \((G, +)\) is always isomorphic to the direct product of \((H, +)\) and \((K, +)\). On the other hand, the division seminear-ring \((G, +, \cdot)\) is left distributive if and only if \((G, \cdot)\) is the direct product of \((H, \cdot)\) and \((K, \cdot)\).

We say that the division seminear-ring defined in Theorem 1 is induced by the ZS-product \( G = H \ast K \). In the finite case, we can prove the converse of Theorem 1.
Theorem 2. Every finite division seminear-ring $D$ is induced by a ZS-product of multiplicative subgroups of $D$.

Proof. Since $(D, +)$ is a finite semigroup, $d + d = d$ for some $d \in D[1]$ and so $x + x = (d + d)d^{-1}x = x$ for all $x \in D$. Now, let $H = \{x \in D : x + 1 = x\}$ and $K = \{x \in D : x + 1 = 1\}$ where 1 is the identity of $(D, \cdot)$. Note that $H \cap K = 1$ and if $x, y \in H$ then $xy + 1 = (x + 1)y + 1 = xy + y = (x + 1)y = xy$: that is, $xy \in H$. Hence, since $(D, \cdot)$ is a finite group, if $x \in H$ then $x^n = 1$ for some $n \geq 1$ and $x^{-1} = x^{n-1} \in H$: that is, $(H, \cdot)$ is a group, and the same holds for $(K, \cdot)$. In particular, if $x, y \in H$ then $xy^{-1} \in H$ and so $xy^{-1} + 1 = xy^{-1}$: that is, $x + y = x$. Hence, if $u, v \in D$, we have $u + v + u = [1 + (vu^{-1} + 1)]u = u$ since $vu^{-1} + 1 \in H$, and so $(D, +)$ is a rectangular band.

To show $D = H \ast K$, let $x \in D$ and note that $x + 1 \in H$, and $x(x + 1)^{-1} + 1 = (x + (x + 1))(x + 1)^{-1} = 1$: that is, $x(x + 1)^{-1} \in K$. Thus, $x = x(x + 1)^{-1}(x + 1) \in KH$ and so, by Lemma 1, $D = H \ast K$. Finally, let $x, y \in D$, and note that $x = x(x + 1)^{-1}(x + 1) \in KH$, $y = y(1 + y)^{-1}(1 + y) \in HK$ and if $z = (x + 1)(1 + y)^{-1} = z(z + 1)^{-1}(z + 1)$ then $(z + 1)(1 + y) = (z + 1)z^{-1}(x + 1)$ where $(z + 1)z^{-1} \in K$. That is, if $\oplus$ denotes the addition induced on $D$ by the ZS-product of $H$ and $K$ then

$$x \oplus y = (z + 1)(1 + y) = (x + 1) + (1 + y)$$

$$= x + [1 + (x + y) + 1] + y = x + y,$$

and this completes the proof.

It can be shown that two finite division seminear-rings are isomorphic if and only if there is a multiplicative group isomorphism between them that preserves the ZS-products of their subgroups, as specified by Theorem 2.

3. Seminear-fields

A right seminear-ring $(F, +, \cdot)$ is a seminear-field if there is $a \in F$ such that $a^2 = a$ and $(F \setminus a, \cdot)$ is a group. In this case, we write $F_a = F \setminus a$ and say that $F$ has base $a$.

If $F = \{a, x\}$ and we define operations $+$ and $\cdot$ on $F$ so that $(F, +)$ is a right zero semigroup and $(F, \cdot)$ is a band with $a$ as an identity then $(F, +, \cdot)$ is a seminear-field in which both $F_a$ and $F_x$ are multiplicative groups. On the other hand, it is easy to check that if $(F, +, \cdot)$ is any seminear-field
with $|F| > 2$ then there is a unique $a \in F$ such that $a^2 = a$ and $(F_a, \cdot)$ is a group.

**Theorem 3.** If $(F, +, \cdot)$ is a seminear-field with base $a$, then $F$ belongs to one of the following categories of seminear-field.

I. $ax = xa = a$ for all $x \in F$, II. $ax = xa = x$ for all $x \in F$, III. $ax = a$ and $xa = x$ for all $x \in F$, IV. $ax = x$ and $xa = a$ for all $x \in F$.

**Proof.** Let $1$ denote the identity of $F_a$ and suppose $a \cdot 1 = a$. If $x \in F_a$ and $ax \neq a$ then there exists $y \in F_a$ with $(ax)y = 1$ and so $a = a(ax)y = 1$, a contradiction. That is, if $a \cdot 1 = a$ then $ax = a$ for all $x \in F$. If $a \cdot 1 \neq a$ then $(a \cdot 1)^2 = a \cdot 1$ implies $a \cdot 1 = 1$ and so $ax = x$ for all $x \in F$. Similarly, we can establish the disjunction: $xa = a$ for all $x \in F$ or $xa = x$ for all $x \in F$, and this proves the result.

Note that any division ring is a category I seminear-field (with $a = 0$), so there is little hope of describing all seminear-fields in category I. On the other hand, those in categories II-IV can be completely characterised.

**Theorem 4.** If $F$ is a category II seminear-field with base $a$, then $(F_a, +, \cdot)$ is a division seminear-ring. Conversely, suppose $(D, +, \cdot)$ is any division seminear-ring and $a \notin D$. Then the operations on $D$ can be extended to $D^* = D \cup a$ so that $(D^*, +, \cdot)$ is a category II seminear-field.

**Proof.** If $x, y \in F_a$ and $x + y = a$ then $1 = a \cdot 1 = x \cdot 1 + y \cdot 1 = a$, a contradiction. Hence, $(F_a, +, \cdot)$ is a division seminear-ring. If $D$ is any division seminear-ring and we extend its operations so that $ax = xa = x$ and $x + a = x + 1$, $a + x = 1 + x$ then it can be checked that $(D^*, +, \cdot)$ is a category II seminear-field.

**Theorem 5.** If $F$ is a category III or IV seminear-field with base $a$, then $|F| = 2$.

**Proof.** If $F$ is category III and $x \in F_a$ then $x^2 = (xa)x = x(ax) = x$ and so $x = 1$. A similar argument works for when $F$ is category IV.

It can be shown that there are only five pairwise non-isomorphic possibilities for the additive structure of category III seminear-fields; namely:

$$
+ \, a \, 1 \\
+ \, a \, 1 \\
a \, a \, a \\
a \, a \, a \\
1 \, a \, 1 \\
a \, a \, 1 \\
1 \, 1 \, 1 \\
1 \, a \, 1
$$
Likewise, it can be shown that there are only three pairwise non-isomorphic possibilities for the additive structure of category IV seminear-fields: namely, the first three tables listed above.

References

