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DIVISION SEMINEAR-RINGS

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We show that any finite division seminear-ring is uniquely determined by the Zappa-Szép product of two multiplicative subgroups, and classify all seminear-fields in three out of four categories.

1. Introduction

In [2] the authors investigated semifields in which the addition and multiplication are both commutative. In [3] the first author extended most of the work in [2] to the non-commutative case: this paper outlines the more significant results in [3].

2. Seminear-rings

We say that $(S, +, \cdot)$ is a **right seminear-ring** if S is a set with two binary operations $+$ and $\cdot$ such that $(S, +)$ and $(S, \cdot)$ are semigroups and the right distributive law holds: $(x + y)z = xz + yz$ for all $x, y, z \in S$. A left seminear-ring is similarly defined, and if S is both a left and a right seminear-ring then it is a semiring. An important example of a right seminear-ring is obtained by starting with an arbitrary semigroup $(S, +)$ and letting $M(S)$ denote the set of all maps from S into itself; if $+$ and $\cdot$ are defined on $M(S)$ as pointwise addition and composition respectively, then $(M(S), +, \cdot)$ is a right seminear-ring which is not left distributive provided $|S| > 1$.

In what follows, the word 'seminear-ring' will mean a 'right seminear-ring'. A **division** seminear-ring is a seminear-ring $(D, +, \cdot)$ in which $(D, \cdot)$ is a group. The set $\mathbf{R}^+$ of positive real numbers with the usual addition and multiplication is a division seminear-ring in which the left distributive

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law holds: that is, it is a division semiring. To obtain a family of right (but not left) division seminear-rings, we need the following notion.

If $(G, \cdot)$ is a group and H, K are subgroups of G, we say G is a **Zappa-Szép (ZS) product** of H and K, written $G = H * K$, if $G = HK$ and $H \cap K = 1$. Note that any direct product is a ZS-product but not conversely. For example, if $G = S_3$ and $H =< (1,2) >, K = A_3$ then $G = H * K$ but S_3 is not a direct product of any of its subgroups. The proof of the following result is straight-forward and so is omitted.

Lemma 1. *If $G = H * K$ for some subgroups H, K of a group G, then $G = K * H$ and*

(a) *for each $x \in G$ there exist unique $h_1, h_2 \in H$ and $k_1, k_2 \in K$ such that $x = h_1 k_1 = k_2 h_2$, and*

(b) *for each $h \in H, k \in K$ there exist unique $h' \in H$ and $k' \in K$ such that $h'k = k'h$.*

The next result provides a way of constructing division seminear-rings which are not division semirings.

Theorem 1. *If $G = H * K$ for some subgroup H, K of group G, then there exists a unique binary operation $+$ on G such that $(G, +, \cdot)$ is a division seminear-ring in which $(G, +)$ is a rectangular band containing H and K as left and right zero subsemigroups respectively and $G = H + K$.*

Proof. Let $x_1, x_2 \in G$. By Lemma 1, we can write $x_1 = k_1 h_1, x_2 = h_2 k_2$ and $h_1' k_2 = k_2' h_1$ for suitable unique elements of H and K. In this case, we define $x_1 + x_2$ to be $h_1' k_2$.

Suppose $x \in G$ and $x = kh_1 = hk_1$. Then, by uniqueness and the definitions, $h_1 = 1 \cdot h_1$, $k_1 = 1 \cdot k_1$ imply that $h_1 + k_1 = hk_1 = x$. Moreover, if $x = h + k = h'k = k'h$ then $h'k = hk_1$ and uniqueness implies $k = k_1$; similarly, $h = h_1$ and we have shown that for each $x \in G$, there are unique $h \in H$, $k \in K$ such that $x = h + k$. In addition, if $hk_1 = kh_1$ and $h'k_2 = k'h_2$ then $(h_1 + k_1) + (h_2 + k_2) = kh_1 + h'k_2$. Hence, if $h''k_2 = k''h_1$ then $(h_1 + k_1) + (h_2 + k_2) = h_1 + k_2$. Now, it is well-known that $H \times K$ under the operation:

$$(h_1, k_1) \otimes (h_2, k_2) = (h_1, k_2)$$

is a rectangular band [1]. And, from the foregoing remarks, $f : G \to H \times K, h + k \to (h, k)$, is an isomorphism from $(G, +)$ onto $(H \times K, \otimes)$. Thus, $(G, +)$ is a rectangular band in which H is a left zero semigroup.

For, if $h \in H$ then $h \cdot 1 = 1 \cdot h$ implies that $h + 1 = h \cdot 1 = h$ and so H is isomorphic under f to $H \times 1$, a left zero subsemigroup of $(H \times K, \otimes)$.

To show that $(G, +, \cdot)$ is right distributive, let $x, y, z \in G$ and write

$$xz = h_1 + k_1 = h_1' k_1 = k_1' h_1$$

$$yz = h_2 + k_2 = h_2' k_2 = k_2' h_2.$$

Then, if $x = h_3 k_3 = k_4 h_4$ and $y = h_5 k_5 = k_6 h_6$, we have $z = x^{-1} k_1' h_1 = y^{-1} h_2' k_2$ and so

$$k_4^{-1} k_1' h_1 = h_4 k_5^{-1} \cdot h_5^{-1} h_2' k_2 = k_7 h_7 \cdot h_5^{-1} h_2' k_2$$

for some h_7, k_7 in G. Since $h_7 h_5^{-1} h_2' \in H$ and $k_7^{-1} k_4^{-1} k_1' \in K$, we conclude that $xz + yz = h_1 + k_2$ and this equals $h_7 \cdot h_5^{-1} h_2' k_2$. But, $x = h_4 + k_3, y = h_6 + k_5$ and so

$$\begin{aligned} (x+y)z &= [(h_4 + k_3) + (h_6 + k_5)] y^{-1} h_2' k_2 \\ &= (h_4 + k_5)(h_5 k_5)^{-1} h_2' k_2 = h_7 h_5^{-1} h_2' k_2 \\ &= xz + yz \end{aligned}$$

since $h_4 + k_5 = h_4' k_5 = k_5' h_4$ implies that $h_4' = h_7$. Finally, to show $+$ is unique, suppose $\oplus$ is another operation for which $(G, \oplus, \cdot)$ has the same properties as $(G, +, \cdot)$. Now, under the stated conditions, $k + h = (1+k)+(h+1) = 1+(k+h)+1 = 1$. Thus, if $h \oplus k = h_1 k_1 = k_2 h_2 = h_2 + k_1$. Then

$$1 = 1 \oplus kh^{-1} \oplus 1 = (h \oplus k) h^{-1} \oplus 1 = k_2 h_2 h^{-1} \oplus 1$$

and so $1 = k_2 \oplus h h_2^{-1} = (k_2 h_2 h^{-1} \oplus 1) h h_2^{-1} = h h_2^{-1}$. Hence, $h = h_2$ and similarly $k = k_1$. So, if $x, y \in G$ satisfy $x = h_1 \oplus k_1 = h_1 + k_1$ and $y = h_2 \oplus k_2 = h_2 + k_2$ then $x \oplus y = h_1 \oplus k_2 = h_1 + k_2 = x + y$, as required.

With the same notation as in Theorem 1, it can be shown that $(G, +)$ is always isomorphic to the direct product of $(H, +)$ and $(K, +)$. On the other hand, the division seminear-ring $(G, +, \cdot)$ is left distributive if and only if $(G, \cdot)$ is the direct product of $(H, \cdot)$ and $(K, \cdot)$.

We say that the division seminear-ring defined in Theorem 1 is **induced** by the ZS-product $G = H * K$. In the finite case, we can prove the converse of Theorem 1.

Theorem 2. *Every finite division seminear-ring D is induced by a ZS-product of multiplicative subgroups of D.*

Proof. Since $(D,+)$ is a finite semigroup, $d+d=d$ for some $d \in D[1]$ and so $x+x=(d+d)d^{-1}x=x$ for all $x \in D$. Now, let $H=\{x \in D : x+1=x\}$ and $K=\{x \in D : x+1=1\}$ where 1 is the identity of $(D,\cdot)$. Note that $H \cap K = 1$ and if $x,y \in H$ then $xy+1=(x+1)y+1=xy+y=(x+1)y=xy$: that is, $xy \in H$. Hence, since $(D,\cdot)$ is a finite group, if $x \in H$ then $x^n=1$ for some $n \geq 1$ and $x^{-1}=x^{n-1} \in H$: that is, $(H,\cdot)$ is a group, and the same holds for $(K,\cdot)$. In particular, if $x,y \in H$ then $xy^{-1} \in H$ and so $xy^{-1}+1=xy^{-1}$: that is, $x+y=x$. Hence, if $u,v \in D$, we have $u+v+u=[1+(vu^{-1}+1)]u=u$ since $vu^{-1}+1 \in H$, and so $(D,+)$ is a rectangular band.

To show $D=H*K$, let $x \in D$ and note that $x+1 \in H$, and $x(x+1)^{-1}+1=(x+(x+1))(x+1)^{-1}=1$: that is, $x(x+1)^{-1} \in K$. Thus, $x=x(x+1)^{-1}(x+1) \in KH$ and so, by Lemma 1, $D=H*K$. Finally, let $x,y \in D$, and note that $x=x(x+1)^{-1}(x+1) \in KH$, $y=y(1+y)^{-1}(1+y) \in HK$ and if $z=(x+1)(1+y)^{-1}=z(z+1)^{-1}(z+1)$ then $(z+1)(1+y)=(z+1)z^{-1}(x+1)$ where $(z+1)z^{-1} \in K$. That is, if $\oplus$ denotes the addition induced on D by the ZS-product of H and K then

$$\begin{aligned} x \oplus y &= (z+1)(1+y)=(x+1)+(1+y) \\ &= x+[1+(x+y)+1]+y=x+y, \end{aligned}$$

and this completes the proof.

It can be shown that two finite division seminear-rings are isomorphic if and only if there is a multiplicative group isomorphism between them that preserves the ZS-products of their subgroups, as specified by Theorem 2.

3. Seminear-fields

A right seminear-ring $(F,+,\cdot)$ is a **seminear-field** if there is $a \in F$ such that $a^2=a$ and $(F\backslash a,\cdot)$ is a group. In this case, we write $F_a=F\backslash a$ and say that F has **base** a.

If $F=\{a,x\}$ and we define operations $+$ and $\cdot$ on F so that $(F,+)$ is a right zero semigroup and $(F,\cdot)$ is a band with a as an identity then $(F,+,\cdot)$ is a seminear-field in which both F_a and F_x are multiplicative groups. On the other hand, it is easy to check that if $(F,+,\cdot)$ is any seminear-field with $|F| > 2$ then there is a **unique** $a \in F$ such that $a^2 \doteq a$ and $(F_a, \cdot)$ is a group.

Theorem 3. *If $(F, +, \cdot)$ is a seminear-field with base a, then F belongs to one of the following categories of seminear-field.*

I. $ax = xa = a$ *for all* $x \in F$, II. $ax = xa = x$ *for all* $x \in F$, III. $ax = a$ *and* $xa = x$ *for all* $x \in F$, IV. $ax = x$ *and* $xa = a$ *for all* $x \in F$.

Proof. Let 1 denote the identity of F_a and suppose $a \cdot 1 = a$. If $x \in F_a$ and $ax \neq a$ then there exists $y \in F_a$ with $(ax)y = 1$ and so $a = a(ax)y = 1$, a contradiction. That is, if $a \cdot 1 = a$ then $ax = a$ for all $x \in F$. If $a \cdot 1 \neq a$ then $(a \cdot 1)^2 = a \cdot 1$ implies $a \cdot 1 = 1$ and so $ax = x$ for all $x \in F$. Similarly, we can establish the disjunction: $xa = a$ for all $x \in F$ or $xa = x$ for all $x \in F$, and this proves the result.

Note that any division ring is a category I seminear-field (with $a = 0$), so there is little hope of describing all seminear-fields in category I. On the other hand, those in categories II-IV can be completely characterised.

Theorem 4. *If F is a category II seminear-field with base a, then $(F_a, +, \cdot)$ is a division seminear-ring. Conversely, suppose $(D, +, \cdot)$ is any division seminear-ring and $a \notin D$. Then the operations on D can be extended to $D^* = D \cup a$ so that $(D^*, +, \cdot)$ is a category II seminear-field.*

Proof. If $x, y \in F_a$ and $x + y = a$ then $1 = a \cdot 1 = x \cdot 1 + y \cdot 1 = a$, a contradiction. Hence, $(F_a, +, \cdot)$ is a division seminear-ring. If D is any division seminear-ring and we extend its operations so that $ax = xa = x$ and $x + a = x + 1$, $a + x = 1 + x$ then it can be checked that $(D^*, +, \cdot)$ is a category II seminear-field.

Theorem 5. *If F is a category III or IV seminear-field with base a, then $|F| = 2$.*

Proof. If F is category III and $x \in F_a$ then $x^2 = (xa)x = x(ax) = x$ and so $x = 1$. A similar argument works for when F is category IV.

It can be shown that there are only five pairwise non-isomorphic possibilities for the additive structure of category III seminear-fields; namely:

$+$	a	1
a	a	a
1	a	1

$+$	a	1
a	a	a
1	1	1

$+$	a	1
a	a	1
1	a	1

+	a	1
a	a	a
1	a	a

+	a	1
a	a	1
1	1	a

Likewise, it can be shown that there are only three pairwise non-isomorphic possibilities for the additive structure of category IV seminear-fields: namely, the first three tables listed above.

References

[1] A. H. Clifford and G. B. Preston, *The Algebraic Theory of Semigroups*, Vol.1, Amer. Math. Soc., Providence, RI.

[2] S. S. Mitchell and P. Sinutoke, *The theory of semifields*, Kyungpook Math. J., 22(1981) 325-347.

[3] J. Hattakosol, *Seminear-fields*, MSc thesis, Department of Mathematics, Chulalongkorn University, 1984.

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