A NOTE ON SEMI-GROUP RINGS WHICH ARE PRE-$p$-RINGS

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In this note we introduce the notion of pre-$p$-rings to semi-group rings and obtain a necessary and sufficient condition under which a semi-group ring is a pre-$p$-ring. In order to obtain this we define a new class of semi-groups called pre-$p$-semigroups. In [1] the authors call an associative and a commutative ring $R$ whose characteristic is $p$ to be a pre-$p$-ring if $x^p y = xy^p$ for every $x, y$ in $R$.

Definition. A commutative semi-group $S$ is called a pre-$p$-semigroup if $x^p y = xy^p$ for all $x$ and $y$ in $S$ and for a fixed prime $p$.

We need the following lemmas to prove our main theorem.

Lemma 1. Let $R$ be a pre-$p$-ring and $S$ a pre-$p$-semigroup then the semi-group ring $RS$ is a pre-$p$-ring.

Proof. Clearly since $R$ and $S$ are pre-$p$-ring and pre-$p$-semigroup respectively the semi-group ring $RS$ is commutative and associative. To prove in $RS$ we have $x^p y = xy^p$ for every $x$ and $y$ in $RS$. Let

$$x = \sum_{i=1}^{n} x_i s_i \quad \text{and} \quad y = \sum_{j=1}^{m} y_j t_j$$

with $x_i, y_j \in R$ and $s_i, t_j \in S$ $1 \leq i \leq n$ and $1 \leq j \leq m$. To prove $x^p y = xy^p$.

Consider

$$x^p y = (\sum_{i=1}^{n} x_i s_i)^p (\sum_{j=1}^{m} y_j t_j)$$
\[ \{ \sum_{i=1}^{n} (x_is_i)^p + p(\sum (\text{terms in products of } x_is_i')s') \sum_{j=1}^{m} y_j t_j. \]

Since \( R \) is a pre-p-ring its characteristic is \( p \), hence the second term in the bracket is zero.

So

\[ x^p y = (\sum_{i=1}^{n} (x_is_i)^p)(\sum_{j=1}^{m} y_j t_j) \]
\[ = (\sum_{i=1}^{n} (x_is_i^p))(\sum_{j=1}^{m} y_j t_j) \]
\[ = \sum_{i,j=1}^{n,m} x_is_i^p y_j s_j^p t_j. \]

Since \( R \) is a pre-p-ring we have

\[ x_is_j^py_j = x_iy_j^p \text{ for every } x_is_j, y_j \in R \]

and since \( S \) is a pre-p-semi-group.

We have \( s_is_j^pt_j = s_jt_j^p \) for every \( s_is_j,t_j \) in \( S \). So

\[ x^p y = \sum_{i,j=1}^{n,m} (x_is_j^p)s_j^pt_j^p. \] (I)

Consider \( xy^p = \sum_{i=1}^{n} x_is_i(\sum_{j=1}^{m} y_j t_j)^p. \)

As before by similar reasoning we get

\[ xy^p = \sum_{i,j=1}^{n,m} x_iy_is_j^pt_j^p. \] (II)

So \( x^p y = xy^p \) from (I) and (II) for every \( x \) and \( y \) in \( RS \). Further \( px = 0 \) for every \( x \) in \( RS \). Hence \( RS \) is a pre-p-ring.

**Lemma 2.** Let \( R \) be a ring with identity and \( S \) a semi-group with identity. If \( RS \) is a pre-p-ring then \( R \) is a pre-p-ring and \( S \) is a pre-p-semi-group.

**Proof.** Since \( RS \) is a pre-p-ring we have \( x^p y = xy^p \) and \( px = 0 \) for every \( x \) and \( y \) in \( RS \). Further \( R \cdot 1 \subseteq RS \). So \( R \) is evidently a pre-p-ring. To
prove $S$ is a pre-p-semi-group. We have $1 \cdot S \subseteq RS$, so $S$ is commutative. Given in $RS$ we have $x^p y = xy^p$ for every $x, y$ in $RS$, so we have for $s$ and $t$ in $S$, $1 \cdot s, 1 \cdot t \in RS$. $(1 \cdot s)^p 1 \cdot t = (1 \cdot s)(1 \cdot t)^p$ for all $s$ and $t$ in $S$. Hence $S$ is a pre-semi-group.

Remark. If we relax the condition, $R$ is any ring without identity then in particular we can have $R$ to be a commutative ring of characteristic $p$ such that $x^p = 0$ for every $x$ in $R$ then $RS$ is a pre-p-ring what ever be $S$.

**Theorem 3.** The semi-group ring $RS$ is a pre-p-ring if and only if $R$ is a pre-p-ring with identity and $S$ a pre-p-semigroup with identity.

It is interesting to know for what groups $G$ the group ring $RG$ is a pre-p-ring.

**Corollary.** The group ring $RG$ is a pre-p-ring if and only if $R$ is a pre-p-ring with identity and $G$ is a commutative torsion group in which the order of every element in $G$ is a divisor of $p - 1$.

**Proof.** For $g \in G \ g^p e = ge^p$ because $G$ is a pre-p-semi-group. Thus $g^p = g$. Hence $g^{p-1} = e$. Thus $(\text{order of } g)/p - 1$. Conversely for all $g, h \in G$, $g^{p-1} = h^{p-1} = e$. Thus $g^p h = gh^p$. Thus $RG$ is a pre-p-semi-group.

**References**


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