A Study on the Multivariate Exponentially Weighted Moving Average Control Charts for Monitoring the Variance-Covariance Matrix

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Abstract

Multivariate exponentially weighted moving average (EWMA) control charts for monitoring the variance-covariance matrix are investigated. Two basic approaches, "combine-accumulate" approach and "accumulate-combine" approach, for using past sample information in the developement of multivariate EWMA control charts are considered. Multivariate EWMA control charts for monitoring the variance-covariance matrix are compared on the basis of their average run length (ARL) performances. The numerical results show that multivariate EWMA control charts based on the accumulate-combine approach are more efficient than corresponding multivariate EWMA control charts based on the combine-accumulate approach.

1. Introduction

Control charts are used to monitor quality variables from a process to detect changes in the parameters of the distribution of these variables. A control chart is maintained by taking samples from a process and plotting in time order on the control chart the relevant statistic computed from the samples.

When control charts are used to monitor production process the main objective is to detect any change in the process that may affect the quality of the output of the process.

There are many situations in which the simultaneous control of two or more quality characteristics is necessary. The original work in multivariate quality control was introduced by Hotelling (1947). Alt (1984) and Jackson (1985) reviewed much of the literature on the multivariate control charts. We will be looking at multivariate EWMA charts for monitoring the variance-covariance matrix.

Suppose that the process of interest has p quality characteristics represented by the random vector $\underline{X} = (X_1, X_2, \cdots, X_p)'$, $p = 2, 3, \cdots$, and $\underline{X}$ has a multivariate normal distribution with mean vector $\underline{\mu}$ and variance-covariance matrix Σ. Let the sample of n observations taken at the i^{th} sampling point be represented by $\underline{X}_i = (\underline{X}_{i1}', \underline{X}_{i2}', \cdots, \underline{X}_{in}')$, where $\underline{X}_{ij}' = (X_{ij1}, X_{ij2}, \cdots, X_{ijp})$ is the j^{th} observation vector among n observation vectors taken at the i^{th} sampling point. Thus $\underline{X}_i$ is an $1 \times np$ vector. It will be assumed that the observation vectors within and between samples are independent. Even though most control charts have this assumption, one should note that this is perhaps not very realistic, since production processes are inherently time dependent.

Suppose that the objective is to monitor Σ where the target value Σ_0 is known. We will consider the case in which the primary objective is to detect changes in the variances, not in the correlation coefficients. Several different control statistics for Σ will be presented since different statistics can be used to describe variability. In the univariate case, the S^2-chart is used to control the variance under the normality assumption. The S^2-chart signals for large values of S^2 or equivalently for large values of $V_i = (n-1)S_i^2/\sigma_0^2$, where $S_i^2 = \sum_{j=1}^{n}(X_{ij} - \bar{X}_i)^2/(n-1)$. One possible multivariate version of V_i is

$$V_i = \sum_{j=1}^{n} (\underline{X}_{ij} - \underline{\bar{X}}_i)' \Sigma_0^{-1} (\underline{X}_{ij} - \underline{\bar{X}}_i) = tr(A_i \Sigma_0^{-1}), \tag{1.1}$$

where $A_i = \sum_{j=1}^{n}(\underline{X}_{ij} - \underline{\bar{X}}_i)(\underline{X}_{ij} - \underline{\bar{X}}_i)'$. When $\Sigma = \Sigma_0$, V_i has a chi-squared distribution with $(n-1)\,p$ degrees of freedom. Hotelling (1947) proposed the use of the Lawley-Hotelling statistic V_i in monitoring the process variance-covariance matrix. The distribution of V_i was studied by Lawley (1938) and Hotelling (1951).

Hui (1980) studied the use of the sample generalized variance in monitoring the process variance-covariance matrix using a statistic $L_i = |A_i/(n-1)|/|\Sigma_0|$. It is known that $(n-1)^{1/2}(L_i - 1)$ is asymptotically normal with mean 0 and variance $2p$ (Anderson (1958)). Another chart can be constructed by using the likelihood ratio statistic for testing $H_0 : \Sigma = \Sigma_0$ vs. $H_1 : \Sigma \neq \Sigma_0$.

In general, if the process shifts from Σ_0 to Σ_1 then it is difficult to obtain the distribution of V_i. Thus, in order to evaluate the properties of the charts for Σ it is necessary to carry out computer simulations.

2. Multivariate EWMA Charts for the Variance-Covariance Matrix

2.1 EWMA Charts based on the combine-accumulate approach

Multivariate EWMA control charts based on the combine-accumulate approach can be constructed by combining multivariate data into a univariate statistic and then forming a univariate EWMA statistic. For the statistic V_i given by (1.1), the multivariate EWMA control chart is based on the control statistic

$$Y_i = (1-\lambda)Y_{i-1} + \lambda V_i, \tag{2.1}$$

where $Y_0 = 0$, $0 < \lambda \leq 1$. This chart signals when $Y_i \geq h_1$.

2.2 EWMA Charts based on the accumulate-combine approach

We propose a multivariate EWMA control chart based on the accumulate-combine approach that accumulates past sample information for each parameter and then combine the separate accumulations into a univariate statistic. Multivariate EWMA control charts based on the accumulate-combine approach can be constructed by forming a univariate statistic from vectors of EWMA's.

In the univariate case, an EWMA control chart for σ^2 can be constructed by using the statistic

$$Y_k = (1-\lambda)Y_{k-1} + \lambda \sum_{j=1}^{n} \left(\frac{X_{kj} - \overline{X}_{k\cdot}}{\sigma_0} \right)^2. \tag{2.2}$$

By repeated substitution in (2.2), it can be shown that

$$Y_k = (1-\lambda)^k Y_0 + \sum_{i=1}^{k} \lambda(1-\lambda)^{k-i} \left\{ \sum_{j=1}^{n} \left(\frac{X_{ij} - \overline{X}_{i\cdot}}{\sigma_0} \right)^2 \right\}, \tag{2.3}$$

$k = 1, 2, \cdots$, and $0 < \lambda \leq 1$.

In the multivariate case, define vectors of EWMA's $\underline{Y}_k = (Y_{k1}, Y_{k2}, \cdots, Y_{kp})'$, where

$$Y_{kl} = (1-\lambda)^k Y_{l0} + \sum_{i=1}^{k} \lambda(1-\lambda)^{k-i} \left\{ \sum_{j=1}^{n} \left(\frac{X_{ijl} - \overline{X}_{i.l\cdot}}{\sigma_{0l}} \right)^2 - (n-1) \right\}, \tag{2.4}$$

$k = 1, 2, \cdots$, $Y_{l0} = 0$, and $0 < \lambda_l \leq 1$, $l = 1, 2, \cdots, p$. A multivariate EWMA chart for

Σ is based on the statistic

$$T_k^2 = \underline{Y}_k' \Sigma_{\underline{Y}_k}^{-1} \underline{Y}_k, \tag{2.5}$$

where $\Sigma_{\underline{Y}_k}$ is the variance-covariance matrix of $\underline{Y}_k$ which will be given in Theorem 1. The vector $\underline{Y}_k$ is one possible multivariate extension of Y_k in (2.3). In general the distribution of $\underline{Y}_k$ is difficult to obtain, but the asymptotic distribution will be obtained for the case in which the process is in control. To simplify notation, let

$$Z_{il} = \sum_{j=1}^{n} \left(\frac{X_{ijl} - \overline{X}_{il\cdot}}{\sigma_{0l}} \right)^2 - (n-1), \text{ and } \underline{Z}_i = (Z_{i1}, Z_{i2}, \cdots, Z_{ip})',$$

for $l = 1, 2, \cdots, p$, $i = 1, 2, \cdots$, then the multivariate EWMA vectors can be expressed as

$$\underline{Y}_k = (1-\lambda)\underline{Y}_{k-1} + \lambda \underline{Z}_k. \tag{2.6}$$

By repeated substitution in (2.6), it can be shown that

$$\underline{Y}_k = \sum_{i=1}^{k} \lambda(1-\lambda)^{k-i} \underline{Z}_i.$$

It is easy to show that

$$E(Z_{il} \mid \mu_l = \mu_{0l}, \sigma_l = \sigma_{il}) = (n-1)\left[\left(\frac{\sigma_{il}}{\sigma_{0l}} \right)^2 - 1 \right], \quad l = 1, 2, \cdots, p.$$

Thus, under the assumption that $\underline{\mu} = \underline{\mu}_0$ and $\Sigma = \Sigma_1$, the expected value of the random vector $\underline{Z}$, denoted by $\underline{\mu}_z$ is

$$\underline{\mu}_z = (n-1)\left[\left(\frac{\sigma_{11}}{\sigma_{01}} \right)^2 - 1, \left(\frac{\sigma_{12}}{\sigma_{02}} \right)^2 - 1, \cdots, \left(\frac{\sigma_{1p}}{\sigma_{0p}} \right)^2 - 1 \right]'.$$

If $\Sigma = \Sigma_0$, then $\underline{\mu}_z = 0$.

Theorem 1. The variance-covariance matrix of $\underline{Y}_k$ when a process is in control and $\underline{Y}_0 = \underline{0}$ is

$$\Sigma_{\underline{Y}_k} = \frac{\lambda}{2-\lambda}[1-(1-\lambda)^{2k}]\Sigma_{\underline{Z}}, \quad \Sigma_{\underline{Z}} = 2(n-1)R^2,$$

where R^2 is used to denote the matrix whose $(l, l')^{th}$ element is the 2^{nd} power of the $(l, l')^{th}$ component of R which is the correlation matrix of $\underline{X} = (X_1, X_2, \cdots X_p)$.

Proof. It is easy to show that from the fact $\underline{Y}_k = \sum_{i=1}^{k}\lambda(1-\lambda)^{k-i}\underline{Z}_i$,

$$\Sigma_{\underline{Y}_k} = \sum_{i=1}^{k} Cov[\lambda(1-\lambda)^{k-i}\underline{Z}_i] = \frac{\lambda}{2-\lambda}[1-(1-\lambda)^{2k}]\Sigma_{\underline{Z}}.$$

When a process is in coctorl, the mean vector and variance-covaiance matrix of $\underline{Z}_i$ is defined as follows: recall that $Z_{il} + (n-1) = \sum_{j=1}^{n}\left(\frac{X_{ijl}-\overline{X}_{il}}{\sigma_{0l}}\right)^2$ has a chi-squared distribution with $(n-1)$ degrees of freedom. Thus $E(Z_{il})=0$ and $Var(Z_{il})=2(n-1)$ for $l=1, 2, \cdots, p$, $i=1, 2, \cdots$. For simplicity, let

$$U_j = \frac{X_{ijl}-\overline{X}_{il}}{\sigma_{0l}}, V_j = \frac{X_{ijl'}-\overline{X}_{il'}}{\sigma_{0l'}}, \text{ and } W_{j'} = \frac{X_{ij'l'}-\overline{X}_{il'}}{\sigma_{0l'}}.$$

Then, they can be expressed that

$$(U_j, V_j) \sim N_2(0, 0, (n-1)/n, (n-1)/n, \rho)$$

and

$$(U_j, W_{j'}) \sim N_2(0, 0, (n-1)/n, (n-1)/n, -\rho/(n-1)).$$

Thus

$$\begin{aligned} Cov(Z_{il}, Z_{il'}) &= Cov[Z_{il}+(n-1), Z_{il'}+(n-1)] \\ &= Cov\left[\sum_{j=1}^{n} U_j^2, \sum_{j=1}^{n} V_j^2\right] \\ &= n\,Cov[U_j^2, V_j^2] + n(n-1)Cov[U_j, W_{j'}]. \end{aligned}$$

By using the moment generating fuction of the bivariate normal distribution, it can be shown that

$$Cov[U_j^2, V_j^2] = 2[Cov(U_j, V_j)]^2,$$

and

$$Cov\,[U_j^2, W_{j'}^2] = 2[Cov(U_j, W_{j'})]^2.$$

Hence

$$Cov\,[Z_{il}, Z_{il}] = 2\frac{(n-1)^2}{n}\rho^2 + 2\frac{(n-1)}{n}\rho^2 = 2(n-1)\rho^2.$$

Therefore $E(\underline{Z}_i) = \underline{0}$ and $\Sigma_{\underline{Z}} = Cov(\underline{Z}_i) = 2(n-1)R^2$.

The following Theorem 2 gives the asymptotic distribution of $\underline{Y}_k$ given by (2.6) when the process is in control.

Theorem 2. Let p-component vectors $\underline{X}_1, \underline{X}_2, \cdots$ be independently identically distributed according to $N_p(\underline{\mu}, \Sigma)$. Then $\{\Sigma_{\underline{Y}_k}^{-1/2}\,\underline{Y}_k,\ k \geq 1\}$ converges in distribution to a multivariate normal distribution with mean vector $\underline{0}$ and variance-covariance matrix I_p as $k \to \infty$, $\lambda \to 0$ and $k\lambda \to 1$.

Proof. Recall that $\underline{Y}_k$ is

$$\underline{Y}_k = \sum_{i=1}^{k} \lambda(1-\lambda)^{k-i}\,\underline{Z}_i.$$

For $k \geq 1$, let

$A_k = \frac{1}{k}\sum_{i=1}^{k} Cov(\lambda(1-\lambda)^{k-i}\,\underline{Z}_i)$,

B_k is the symmetric, positive definite matrix satisfying $B_k^2 = A_k^{-1}$,

γ_k = the smallest eigenvalue of A_k.

By the corollaries 18.2 and 18.3 of Bhattacharya and Rao (1975), if

$$\Theta_k(\lambda) = k^{-3/2}\sum_{i=1}^{k} E\|B_k\lambda(1-\lambda)^{k-i}\,\underline{Z}_i\|^3 \to 0$$

as $k \to \infty$, $\lambda \to 0$, and $k\lambda \to 1$.

then

$$\frac{1}{\sqrt{k}} B_k \sum_{i=1}^{k} \lambda(1-\lambda)^{k-i}\,\underline{Z}_i \xrightarrow{d} N_p(\underline{0}, I), \text{ as } k \to \infty,\ \lambda \to 0,\ k\lambda \to 1.$$

The inequality given (17.63) of Bhattacharya and Rao (1975) is

$$\|B_k \underline{Z}_i\| \leq \gamma_k^{-1/2} \|\underline{Z}_i\|, \qquad 1 \leq i \leq k.$$

and this gives

$$\|B_k \underline{Z}_i\|^3 \leq \gamma_k^{-3/2} \|\underline{Z}_i\|^3, \qquad 1 \leq i \leq k.$$

Thus

$$\begin{aligned} \Theta_k(\lambda) &= k^{-3/2} \sum_{i=1}^{k} E\|B_k \lambda(1-\lambda)^{k-i} \underline{Z}_i\|^3 \\ &= k^{-3/2} \lambda^3 \sum_{i=1}^{k} (1-\lambda)^{3(k-i)} E\|B_k \underline{Z}_i\|^3 \\ &\leq k^{-3/2} \lambda^3 \sum_{i=1}^{k} (1-\lambda)^{3(k-i)} \gamma_k^{-3/2} E\|\underline{Z}_i\|^3. \end{aligned} \tag{2.7}$$

Now

$$A_k = \frac{1}{k} \Sigma_{\underline{Y}_k} = \frac{1}{k} \frac{\lambda}{(2-\lambda)} [1-(1-\lambda)^{2k}] \Sigma_{\underline{Z}}.$$

Let γ be the smallest eigenvalue of $\Sigma_{\underline{Z}}$, then the samallest eigenvalue of A_k can be expressed as

$$\gamma_k = \{ \frac{1}{k} \frac{\lambda}{(2-\lambda)} [1-(1-\lambda)^{2k}] \} \gamma.$$

Thus, the right hand side of inequality (2.7) is less than or equal to

$$[\{ \frac{1}{k} \frac{\lambda}{(2-\lambda)} [1-(1-\lambda)^{2k}] \} \gamma]^{-3/2} \lambda^3 \sum_{i=1}^{k} (1-\lambda)^{3(k-i)} E\|\underline{Z}_i\|^3. \tag{2.8}$$

By using the inequality given by Chung (1974, p. 48), it is easy to show that

$$E\|\underline{Z}_i\|^3 = E(\sum_{l=1}^{p} Z_{il}^2)^{3/2} \leq \sqrt{p} \sum_{l=1}^{p} E|Z_{il}|^3 = p^{3/2} E|Z_{il}|^3, l = 1, 2, \cdots, p.$$

Let $m_3 = E|Z_{il}|^3 < \infty$, $l=1, 2, \cdots, p$. Thus, the quantity (2.8) is less than or equal to

$$(\frac{p}{\gamma})^{3/2} m_3 \frac{\sqrt{\lambda}(1-(1-\lambda)^{3k})}{(\lambda^2-3\lambda+3)} [\frac{2-\lambda}{1-(1-\lambda)^{2k}}]^{3/2} \to 0.$$

as $k \to \infty$, $\lambda \to 0$, and $k\lambda \to 1$.

Therefore

$$\frac{1}{\sqrt{k}} B_k \sum_{i=1}^{k} \lambda(1-\lambda)^{k-i} \underline{Z}_i = \Sigma_{\underline{Y}_k}^{-1/2} \underline{Y}_k, \xrightarrow{d} N_p(0, I),$$

as $k \to \infty$, $\lambda \to 0$, and $k\lambda \to 1$.

Corollary. $\{ T_k^2,\ k \geq 1\}$ converges in distribution to a chi-squared distribution with p degrees of freedom as $k \to \infty$, $\lambda \to 0$, and $k\lambda \to 1$.

Proof. Recall that the control statistic T_k^2 is

$$T_k^2 = \underline{Y}_k' \Sigma_{\underline{Y}_k}^{-1} \underline{Y}_k,$$

which can be expressed as

$$T_k^2 = (\Sigma_{\underline{Y}_k}^{-1/2} \underline{Y}_k)' (\Sigma_{\underline{Y}_k}^{-1/2} \underline{Y}_k).$$

By Theorem 2 and the corollary of Serfling (1980, p. 25),

$$T_k^2 = \underline{Y}_k' \Sigma_{\underline{Y}_k}^{-1} \underline{Y}_k \xrightarrow{d} \chi^2(p), \text{ as } k \to \infty, \lambda \to 0, k\lambda \to 1.$$

The multivariate EWMA chart signals that the process is out-of-control whenever $T_k^2 \geq h_2$. The ARL performance of the multivariate EWMA chart based on accumulate-combine approach can not be modeled as a simple stationary Markov chain as described in Brook and Evans (1972). A simulation to obtain the ARL values and parameter h_2 was used.

3. Numerical Results

The following control procedures will be compared on the basis of their ARL performances.

1. Multivariate EWMA chart with control statistic given by (2.1).
2. Multivariate EWMA chart with control statistic given by (2.5)

The performance of the charts for monitoring a variance-covariance matrix depends on the value Σ. It is not possible to investigate all of the different ways in which Σ could change, but the following types of shifts are considered:

(V1) all variances and covariances are changed by a constant factor i.e. $\Sigma_1 = c\Sigma_0$.

(V2) one variance increased to $c1\sigma_{0ii}$ and the other variances are remained on target.

(V3) approximately half of the variances and covariances are changed by a constant and there are no shifts in the rest.

When comparing control charts, some kinds of standard for comparison is necessary. The charts are matched for ARL when the process is in control. This enables the performance to be evaluated when the process has shifted away from its target value. In our computation, the ARL in control was fixed to be 200 and the sample size used for each sample observation was 5. It is assumed that the correlation coefficient ρ is the same for all variables. Table 1 and 2 gives the values of h_1 and h_2, respectively for $p=2$-5 and λ when the ARL at $\Sigma=\Sigma_0$ is approximately 200. ARL values and parameters h_1, h_2 were calculated by using Markov chain approach or 10,000 simulations. For $p=2$ and three different correlation coefficients $\rho=0.0$, 0.5, 0.8, Tables 3-5 give ARL values. As shown in Tables 3-5, for the multivariate EWMA control charts based on the accumulate-combine approach, smaller values of λ are more effective in detecting all shifts in Σ for $p=2$. Also, Tables 3-5 show that for the multivariate EWMA control charts based on combine-accumulate approach, small values of λ are more effective in detecting small and moderate shifts, and large values of λ are more effective in detecting large shifts. From Tables 3-5, $\lambda=0.1-0.5$ seems to work well over a range of different shifts. For $p=5$, $\rho=0.5$, Table 6 gives ARL values. The results in Tables 3-6 show that multivariate EWMA chart based on accumulate-combine approach is better than multivariate EWMA chart based on combine-accumulate approach.

4. Conclusions

Numerical results show that ARL of the multivariate EWMA control charts based on accumulate-combine approach decreases as λ decreases. Thus small values of λ may be best for practical applications. The multivariate EWMA control charts based on accumulate-combine approach are more effective in detecting all shifts in Σ in terms of ARL than the multivariate EWMA control charts based on combine-accumulate approach.

〈 Table 1 〉 Values of h_1 for Various Values of λ and p when the ARL at $\Sigma=\Sigma_0$ is 200

p	λ=0.7	λ=0.5	λ=0.3	λ=0.1	λ=0.05
2	17.9326	15.4050	12.8841	10.0350	9.0624
3				14.4509	
4				18.8011	
5				23.1094	

〈 Table 2 〉 Values of h_2 for Various Values of λ and p when the ARL at $\Sigma=\Sigma_0$ is approximately 200

p	ρ	λ=0.3	λ=0.1	λ=0.05
2	0.0	14.0154	9.4105	7.7675
2	0.5	14.3502	9.5307	7.8020
2	0.8	14.3502	9.5307	7.8020
3	0.5			10.0917
4	0.5			12.1742
5	0.5			14.0632

〈 Table 3 〉 ARL Values for Multivariate EWMA Charts for Monitoring the Variance-Covariance Matrix (p=2, ρ=0.0)

	λ=0.05		λ=0.10		λ=0.03		λ=0.50	λ=0.70
shift	A-C	C-A	A-C	C-A	A-C	C-A	A-C C-A	A-C C-A
c=1.00	199.5	200.0	199.6	200.0	199.5	200.0	200.0	200.0
c=1.21	21.7	50.5	24.4	36.5	34.4	33.1	37.8	43.0
c=1.69	4.1	22.4	4.6	13.7	6.0	7.9	7.4	7.8
c=2.56	2.1	12.1	2.2	7.1	2.7	3.6	2.9	2.7
$c1$=1.21	38.4	78.0	42.9	64.8	60.7	67.7	75.5	82.8
$c1$=1.69	7.7	36.5	8.5	24.0	11.7	17.3	18.5	20.5
$c1$=2.56	2.9	20.6	3.2	12.5	3.9	7.0	6.2	6.3

In Tables 3-6,

A-C stands for EWMA chart based on the accumulate-combine approach

C-A stands for EWMA chart based on the combine-accumulate approach

〈 Table 4 〉 ARL Values for Multivariate EWMA Charts for Monitoring the Variance-Covariance Matrix (p=2, ρ=0.5)

shift	λ=0.05 A-C	λ=0.05 C-A	λ=0.10 A-C	λ=0.10 C-A	λ=0.03 A-C	λ=0.03 C-A	λ=0.50 A-C C-A	λ=0.70 A-C C-A
c=1.00	200.6	200.0	200.5	200.0	200.2	200.0	200.0	200.0
c=1.21	24.9	50.5	27.8	36.5	38.6	33.1	37.8	43.0
c=1.69	4.7	22.4	5.3	13.7	6.9	7.9	7.4	7.8
c=2.56	1.9	12.1	2.1	7.1	2.4	3.6	2.9	2.7
$c1$=1.21	37.4	77.1	42.5	63.8	61.7	66.5	73.9	80.7
$c1$=1.69	7.4	35.5	8.2	23.2	11.4	16.4	17.2	19.0
$c1$=2.56	2.8	29.5	3.1	11.8	3.8	6.5	5.7	5.7

〈 Table 5 〉 ARL Values for Multivariate EWMA Charts for Monitoring the Variance-Covariance Matrix (p=2, ρ=0.8)

shift	λ=0.05 A-C	λ=0.05 C-A	λ=0.10 A-C	λ=0.10 C-A	λ=0.03 A-C	λ=0.03 C-A	λ=0.50 A-C C-A	λ=0.70 A-C C-A
c=1.00	200.1	200.0	201.7	200.0	200.8	200.0	200.0	200.0
c=1.21	28.5	50.5	31.5	36.5	42.0	33.1	37.8	43.0
c=1.69	5.4	22.4	6.0	13.7	7.9	7.9	7.4	7.8
c=2.56	2.1	12.1	2.3	7.1	2.7	3.6	2.9	2.7
$c1$=1.21	27.9	73.9	33.0	60.3	52.0	61.4	68.0	73.2
$c1$=1.69	5.4	31.6	6.1	20.2	8.3	13.3	13.2	14.2
$c1$=2.56	2.2	16.0	2.4	9.6	2.8	5.1	4.3	4.0

〈 Table 6 〉 ARL Values for Multivariate EWMA Charts for Monitoring the Variance-Covariance Matrix (p=5, ρ=0.5)

Shift	λ=0.05 A-C	λ=0.10 C-A	shift	λ=0.05 A-C	λ=0.10 C-A
c=1.00	200.4	200.0	$c1$-3=1.21	22.7	40.8
c=1.21	18.6	27.8	$c1$-3=1.44	7.5	22.8
c=1.69	3.2	11.8	$c1$-3=1.69	4.1	16.4
c=2.56	1.4	6.3	$c1$-3=1.96	2.7	12.8
$c1$=1.21	46.3	90.2	$c1$-3=2.25	2.0	10.4
$c1$=1.69	8.5	35.7	$c1$-3=2.56	1.7	8.8
$c1$=2.56	3.2	18.6			

$c1$-3 represents that 3 variances among 5 variances are changed by a constant and the other variances are remained on target

References

[1] Alt, F. B.(1984), *Multivariate Control Charts*, in Encyclopedia of Statistical Sciences, eds. S. Kotz and N. L. Johnson, Wiley, New York.

[2] Anderson, T. W.(1958), *An Introduction to Multivariate Analysis*, Wiley, New York.

[3] Bhattacharya, M. and Rao, R. R.(1975), *Normal Approximation and Asymptotic Expansions*, Wiley, New York.

[4] Brook, D. and Evans, D. A.(1972), "An Approach to the Probability Distribution of CUSUM Run Length," *Biometrika*, Vol. 59, pp. 539–549.

[5] Chung, K. L.(1974), *A Course In Probability*, Academic Press, New York.

[6] Crosier, R. B.(1988), "Multivariate Generalizations of Cumulative Sum Quality-Control Schemes," Technometrics, Vol. 30, pp. 291–303.

[7] Hotelling, H.(1974), *Multivariate Quality Control. Techniques of Statistical Analysis*, McGraw-Hill, pp. 111–184.

[8] Hotelling, H.(1951), "A Generalized T Test and Measure of Multivariate Dispersion," *Proc. Second Berkely Symp. Math. Statist, Prob., Berkely, U. of California Press*, pp. 23–42.

[9] Jackson, J. S.(1985), "Multivariate Quality Control," *Communications in Statistics-Theory and Methods*, Vol. 14, pp.2657–2688.

[10] Lawley, D. N.(1938), "A Generalization of Fisher's Z-Test," *Biometrika*, Vol. 30, pp. 180–187.

[11] Serfling R. J.(1980), *Approximation Theorems of Mathematical Statistics*. Weley, New York.