Multivariate Control Charts for Means and Variances with Variable Sampling Intervals

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Abstract

Several sample statistics to simultaneously monitor both means and variances for multivariate quality characteristics under multivariate normal process are proposed. Performances of multivariate Shewhart schemes and cumulative sum (CUSUM) schemes are evaluated for matched fixed sampling interval(FSI) and variable sampling interval(VSI) feature. Numerical results show that multivariate CUSUM charts are more efficient than Shewhart charts for small or moderate shifts and VSI feature is more efficient than FSI feature.

1. Introduction

Many situations in industrial quality control involve a vector of measurements of two or more related quality characteristics rather than a single characteristic. When the quality characteristics are correlated, one could obtain better sensitivity by using multivariate control chart than seperate control charts for each of the quality characteristics. Before recent years, the use of multivariate quality control techniques was hampered by the lack of adequate computaional resources. Jackson(1959) and Ghare and Torgersen(1968) presented multivariate Shewhart control chart based on Hotelling's T^2 statistic. Woodall and Ncube(1985) considered a single multivariate CUSUM procedure for controlling the mean of multivariate normal process. They described how a p dimensional multivariate normal process can be monitored by using p two-sided univariate CUSUM charts.

One traditional practice in using a control chart is to take samples from the process at FSI and properties of control charts have been developed when the sampling interval between samples is fixed. In recent years, application of VSI control charts has become quite frequent and several papers have been published about them in which the sampling interval is varied as a function of what is observed from the process. The basic idea of a VSI control chart is that the time interval should be short if there is "an indication" of a process change and long if there is no indication of a process change.

The ability of a control chart to detect process changes is determined by the length of time required for the chart to signal. In FSI chart, the run length(RL) is defined as the random number of samples required for the chart to signal and the average run length(ARL) is the expected value of the RL. Therefore, the expected time to signal is simply the product of the ARL and the length of the fixed sampling interval in FSI chart.

In VSI chart, the sampling intervals are random variables and the time required to signal is not the product of the number of samples and a fixed sampling interval. Thus, for the performance of a VSI chart, it is necessary to keep track of both the time to signal(TS) and the number of samples to signal(NSS). Following the definition of Reynolds et al.(1988), the NSS is the number of samples taken from the start of the process to the time when the chart signals and the average number of samples to signal(ANSS) is the expected value of the NSS. Also, they defined that the TS is the time from the start of the process to the time when the chart signals and the average time to signal(ATS) is the expected value of the TS. The NSS has the same definition as the RL but it seems preferable to use NSS because it is more descriptive.

Reynolds and Arnold(1989) and Reynolds(1989) investigated the theoretical aspects of one-and two-sided VSI Shewhart charts in detail and showed that the use of two sampling intervals spaced as apart as possible is optimal. One disadvantage of VSI scheme is that frequent switching between different sampling intervals requires more cost and effort to administer the process than corresponding FSI scheme. Amin and Letsinger(1991) described general procedures for combining the VSI feature and examined the switching behavior and runs rules for switching between the different sampling intervals. They also presented that average number of switches to signal (ANSW) of the CUSUM and EWMA procedures exists far fewer than that of the Shewhart procedure.

Most of the studies on multivariate control charts have been concentrated on monitoring mean vector of multivariate normal process. In this paper, we present a single multivariate control scheme to simultaneously monitor both the means and variances of the multivariate normal process.

2. Constructing Multivariate Sample Statistics for Control Charts

Assume that the process of interest has $p(p \geq 2)$ related quality characteristics represented by the random vector $\underline{X} = (X_1, X_2, \cdots, X_p)'$ and we obtain a sequence of random vectors $\underline{X}_1, \underline{X}_2, \cdots$, where $\underline{X}_i = (\underline{X}_{i1}', \cdots, \underline{X}_{in}')'$ is a sample of observations at the sampling poing $i (i=1, 2, \cdots)$ and $\underline{X}_{ij} = (X_{ij1}, \cdots, X_{ijp})'$. It will be assumed that the sequential observation vectors between and within samples are independent and identically distributed.

In this paper, we assume that process quality variables have a multivariate normal distribution. Hence, the distribution of $\underline{X}$ is indexed by a set of parameters $\underline{\theta} = (\underline{\mu}, \Sigma)$ where $\underline{\mu}$ is the mean vector and Σ is the covariance matrix of $\underline{X}$. Let $\underline{\theta}_0 = (\underline{\mu}_0, \Sigma_0)$ be the target values for $\underline{\theta}$ of p related quality characteristics and $\underline{\theta}_0$ is represented as

$$\underline{\mu}_0 = \begin{bmatrix} \mu_{10} \\ \mu_{20} \\ \vdots \\ \mu_{p0} \end{bmatrix} \quad \text{and} \quad \Sigma_0 = \begin{bmatrix} \sigma_{10}^2 & \rho_{120}\sigma_{10}\sigma_{20} & \cdots & \rho_{1p0}\sigma_{10}\sigma_{p0} \\ \rho_{120}\sigma_{10}\sigma_{20} & \sigma_{20}^2 & \cdots & \rho_{2p0}\sigma_{20}\sigma_{p0} \\ \vdots & \vdots & \cdots & \vdots \\ \rho_{1p0}\sigma_{10}\sigma_{p0} & \rho_{2p0}\sigma_{20}\sigma_{p0} & \cdots & \rho_{p0}^2 \end{bmatrix},$$

where the target covariance of X_r and X_s is $\sigma_{rs0} = \rho_{rs0}\sigma_{r0}\sigma_{s0}$ for $r, s=1, 2, \cdots, p$. In this paper, we assume that the parameters $\underline{\mu}_0$ and Σ_0 are known. For simplicity, we also assume that $\mu_{r0}=0$, $\sigma_{r0}^2=1.0$, $\rho_{rs0}=0.3$ for $r, s=1, 2, \cdots, p$.

To control both μ and σ^2 of a quality characteristic, Reynolds and Ghosh(1981) proposed an obvious method to use the statistic UV_i and the chart signals if

$$UV_i = \sum_{j=1}^{n}(X_{ij} - \mu_0)^2 / \sigma_0^2 \geq \chi_{1-\alpha}^2(n).$$

Extending the sample statistic UV_i to multivariate case, we obtain the sample statistic D_i for multivariate control chart

$$D_i = \sum_{j=1}^{n}(\underline{X}_{ij} - \underline{\mu}_0)'\Sigma_0^{-1}(\underline{X}_{ij} - \underline{\mu}_0). \tag{2.1}$$

By simple calculation, we have

$$D_i = Z_i^2 + V_i, \tag{2.2}$$

where $Z_i^2 = n(\underline{\bar{X}}_i - \underline{\mu}_0)'\Sigma_0^{-1}(\underline{\bar{X}}_i - \underline{\mu}_0)$, $V_i = tr(A_i\Sigma_0^{-1})$, $A_i = \sum_{j=1}^{n}(\underline{X}_{ij} - \underline{\bar{X}}_i)(\underline{X}_{ij} - \underline{\bar{X}}_i)'$

and $\overline{X}_i$ is the sample mean vector of the n observations on occasion i.

If the process is in-control, D_i has a chi-squared distribution with np degrees of freedom. Alt(1982) used Z_i^2 for controlling the mean vector $\underline{\mu}$. And we use V_i for monitoring the dispersion matrix Σ. Note that Z_i^2 and V_i are independent, $Z_i^2 \sim \chi^2(p)$ and $V_i \sim \chi^2((n-1)p)$ when $\underline{\mu}=\underline{\mu}_0$ and $\Sigma=\Sigma_0$.

When the process has shifted to $\underline{\mu}$ from the target $\underline{\mu}_0$, D_i and Z_i^2 has a non-central chi-squared distribution with np and p degrees of freedom, respectively, with noncentrality parameter $\tau^2=n(\underline{\mu}-\underline{\mu}_0)'\Sigma_0^{-1}(\underline{\mu}-\underline{\mu}_0)$.

Another control chart for both $\underline{\mu}$ and Σ can be constructed by using both of the likelihood ratio statistic for testing $H_0: \underline{\mu}=\underline{\mu}_0$ vs $H_1: \underline{\mu}\neq\underline{\mu}_0$ where Σ_0 is known, and $H_0: \Sigma=\Sigma_0$ vs $H_1: \Sigma\neq\Sigma_0$ where $\underline{\mu}_0$ is known. Likilihood ratio λ_1 of the ith sample for testing $H_0: \underline{\mu}=\underline{\mu}_0$ vs $H_1: \underline{\mu}\neq\underline{\mu}_0$ where Σ_0 is known can be expressed as

$$\lambda_1 = \exp\left[-\frac{n}{2}(\overline{X}_i-\underline{\mu}_0)'\Sigma_0^{-1}(\overline{X}_i-\underline{\mu}_0)\right].$$

Let TM_i be $-2\ln\lambda_1$, then

$$TM_i = n(\overline{X}_i-\underline{\mu}_0)'\Sigma_0^{-1}(\overline{X}_i-\underline{\mu}_0). \tag{2.3}$$

Similarily, likelihood ratio λ_2 for testing $H_0: \Sigma=\Sigma_0$ vs $H_1: \Sigma\neq\Sigma_0$ where $\underline{\mu}_0$ is known can be expressed as

$$\lambda_2 = n^{-\frac{np}{2}} \cdot |A_i\Sigma_0^{-1}|^{\frac{n}{2}} \exp\left[-\frac{1}{2}tr(\Sigma_0^{-1}A_i)+\frac{1}{2}np\right].$$

Let TV_i be $-2\ln\lambda_2$, then

$$TV_i = tr(A_i\Sigma_0^{-1})-n\ln|A_i|+n\ln|\Sigma_0|+np\ln n-np. \tag{2.4}$$

Thus, the statistics TM_i and TV_i can be used as sample statistics for monitoring $\underline{\mu}$ and Σ, respectively. Since the statistic TM_i has a chi-squared distribution with p degrees of freedom, the percentage points of TM_i can be obtaind from the chi-square distribution. But it is difficult to obtain the distribution of TV_i. Therefore, the percentage points of TV_i can be obtained by simulations.

Thus, we propose the following three different sample statistics for simultaneously monitoring both $\underline{\mu}$ and Σ :

1. $D_i = \sum_{j=1}^{n} (\underline{X}_{ij} - \underline{\mu}_0)' \Sigma_0^{-1} (\underline{X}_{ij} - \mu_0)$
2. simultaneous use of (Z_i^2, V_i)
3. simultaneous use of (TM_i, TV_i).

3. Multivariate Shewhart Chart

Shewhart control charts are widely used to display sample data from a process for purposes of determining whether a process is in-control, for bringing an out-of-control process into in-control, and for monitoring a procsess to make sure that it stays in-control. When the chart signals that an assignable cause is present, then a rectifying action is taken to remove the cause and bring the process back into in-control. A Shewhart chart has a good ability to detect large changes in monitored parameter quickly and is easy to implement the process. However, the Shewhart chart is slow to signal small or moderate changes in the parameter.

Let q be the probability that a control statistic falls out-of-control limits and d be the sampling interval for FSI scheme. Then, for FSI control charts, the time required to signal T is dN where N is NSS. Since N is geometrically distributed with parameter q, when the process is in-control, the ATS is given by

$$E(T) = dE(N) = d/q,$$

and the variance of T is

$$Var(T) = d^2 (1-q)/q^2.$$

For VSI control charts, if we use a finite number of interval lengths $d_1, d_2, \cdots, d_\eta$ where $d_1 < d_2 < \cdots < d_\eta$, these possible lengths might be determined by the physical considerations. Let the in-control region be partitioned into η regions $I_1, I_2, \cdots, I_\eta$ where I_i is the region in which the sampling interval d_i is used. Thus the sampling interval used between $\underline{X}_i$ and $\underline{X}_{i+1}$ is represented by a sampling interval function $d(\underline{X}_i)$. In VSI Shewhart chart, the ANSS is

$$E(N) = 1/q,$$

and the variance of N is

$$Var(N) = (1-q)/q^2.$$

If we let T be TS, and R_i, be sampling interval used before the ith sample is taken, then $T = \sum_{i=1}^{N} R_i$. Note that the distribution of each R_i is the conditional distribution of $d(\underline{X}_i)$ given that there is no signal at the previous sample.

For simplicity, we assume that the chart is started at time 0 and that R_1, the interval used before the first sample, is a fixed constant, say d_0. Then the ATS can be expressed as

$$E(T) = d_0 + E(N-1)E(R_i) = d_0 + (ANSS-1) \cdot E(R_i),$$

assuming that the process parameters remain constant. Reynolds et al.(1988) showed that the expected value of R_i is

$$E(R_i) = \sum_{j=1}^{\eta} d_j p_j / (1-q),$$

where $p_j = P(d(\underline{X}_i) = d_j)$ and $\sum_{j=1}^{\eta} p_j = 1-q < 1$.

Thus, the ATS is given as

$$E(T) = d_0 + \sum_{j=1}^{\eta} d_j p_j / q, \tag{3.1}$$

and we can obtain

$$Var(T) = \sum_{j=1}^{\eta} d_j^2 p_j / q + (\sum_{j=1}^{\eta} d_j p_j)^2 / q^2. \tag{3.2}$$

A multivariate Shewhart control chart based on sample statistic D_i signals when

$$D_i \geq \chi^2_{1-\alpha}(np).$$

A multivariate chart based on sample statistics (Z_i^2, V_i) uses separate charts for μ and Σ, and signals if one of the two charts for $\underline{\mu}$ or Σ signals when

$$Z_i^2 \geq \chi^2_{1-\alpha_{\underline{\mu}}}(p) \quad \text{or} \quad V_i \geq \chi^2_{1-\alpha_\Sigma}((n-1)p).$$

The overall false alarm probability of the chart based of (Z_i^2, V_i) is $1-(1-\alpha_{\underline{\mu}})(1-\alpha_\Sigma)$ where the signal probabilities of the chart for $\underline{\mu}$ and Σ are $\alpha_{\underline{\mu}}$ and α_Σ, respectively. The percentage points of D_i, Z_i^2 and V_i can be obtained from the chi-square distributions.

And a multivariate chart based on the statistics (TM_i, TV_i) uses the separate chart for μ and Σ, and signals if one of the two charts for μ or Σ signals when

$$TM_i \geq \chi^2_{1-\alpha_\mu}(p) \quad \text{or} \quad TV_i \geq h_{TM},$$

where the parameter h_{TM} can be obtained by simulation.

For the VSI multivariate Shewhart chart based on D_i with two sampling intervals, suppose that the sampling interval ;

d_1 is used when $D_i \in (g_D, h_D]$,

d_2 is used when $D_i \in [0, g_D)$.

Similarly, for the VSI scheme based on (Z_i^2, V_i), suppose that ;

d_1 is used when $Z_i^2 \in (g_{Z2}, h_{Z2}]$ or $V_i \in (g_V, h_V]$,

d_2 is used when $Z_i^2 \in (0, g_{Z2}]$ and $V_i \in (0, g_V]$.

And for the VSI scheme based on (TM_i, TV_i), suppose that ;

d_1 is used when $TM_i \in (g_{TM}, h_{TM}]$ or $TV_i \in (g_{TV}, h_{TV}]$,

d_2 is used when $TM_i \in (0, g_{TM}]$ and $TV_i \in (0, g_{TV}]$.

The parameters g_D, h_D, g_{Z2}, h_{Z2}, g_V, h_V, g_{TM} and h_{TM} can be obtained from the desired percentage points of chi-square distribution to satisfy a desired ATS and ANSS. And the parameters g_{TV} and h_{TV} can be obtained by simulation.

Result 3.1 Let $\underline{X}_{ij} = (X_{ij1}, X_{ij2}, \cdots, X_{ijp})'$ be distributed according to $N_p(\underline{\mu}_0, \Sigma_0)$ and $\underline{X}_{ij}$'s be independent. Assume that VSI multivariate control scheme based on D_i in (2.1) is used and two sampling intervals are used as stated above. If the parameters of the distribution is shifted as $N(\underline{\mu}, c\Sigma_0)$ where c is a constant, then

$$ANSS = 1 / [1-F(h_D / c)] \tag{3.3}$$

and

$$ATS = d_0 + [d_1[F(h_D/c)-F(g_D/c)] + d_2 F(g_D/c)] / [1-F(h_D/c)], \tag{3.4}$$

where $F(\cdot)$ is chi-squared distribution function with np degrees of freedom and noncentrality parameter $n(\underline{\mu}-\underline{\mu}_0)'\Sigma_0^{-1}(\underline{\mu}-\underline{\mu}_0)/c$.

4. Multivariate Cusum Chart

The basic Shewhart chart, although simple to understand and apply, uses only the information in the current sample and is thus relatively inefficient in detecting small changes in control parameters. Modifications such as the use of supplementary runs rules improve the efficiency of Shewhart chart somewhat but other approaches which take full advantages of the information in past samples are needed. CUSUM charts are often used instead of standard Shewhart chart when detection of small changes in a process parameter is important.

The most direct and obvious method of replacing the multivariate Shewhart chart by a CUSUM procedure for simultaneously monitoring $\underline{\mu}$ and Σ is to form a CUSUM based on the statistics D_i, (Z_i^2, V_i) or (TM_i, TV_i).

A multivariate CUSUM for $\underline{\mu}$ and Σ based on the statistic D_i at the ith sample is

$$Y_{D,i} = \max\{Y_{D,i-1}, 0\} + \{D_i - k_D\}, \tag{4.1}$$

where $Y_{D,0} = w \cdot I_{(w \geq 0)}$, $k_D \geq 0$. This chart signals whenever $Y_{D,i} \geq h_D$.

For the VSI multivariate CUSUM chart, suppose that the sampling interval ;

d_1 is used when $Y_{D,i} \in (g_D, h_D]$,

d_2 is used when $Y_{D,i} \in (-k_D, g_D]$,

where $-k_D < g_D \leq h_D$. When the parameters are on-target or mean vector $\underline{\mu}$ has shifted, the properties of the multivariate CUSUM chart based on D_i can be evaluated by the Markov chain or integral equation approach. When the process shifts in Σ, or $\underline{\mu}$ and Σ, the properties of the chart can be evaluated by simulation.

For simultaneous use of (Z_i^2, V_i) given in (2.2), the multivariate CUSUM chart for $\underline{\mu}$ at the ith sample is

$$Y_{Z^2,i} = \max\{Y_{Z^2,i-1}, 0\} + (Z_i^2 - k_{Z^2}) \tag{4.2}$$

and for Σ at the ith sample is

$$Y_{V,i} = \max\{Y_{V,i-1}, 0\} + (V_i - k_V), \tag{4.3}$$

where $Y_{Z^2,0} = w_1 \cdot I_{(w_1 \geq 0)}$, $Y_{V,0} = w_2 \cdot I_{(w_2 \geq 0)}$, $k_{Z^2} \geq 0$ and $k_V \geq 0$. This chart signals whenever $Y_{Z^2,i} \geq h_{Z^2}$ or $Y_{V,i} \geq h_V$.

For the VSI scheme, suppose that the sampling interval ;

d_1 is used when $Y_{Z^2,i} \in (g_{Z^2}, h_V]$ or $Y_{V,i} \in (g_V, h_V]$,

d_2 is used when $Y_{Z^2,i} \in (-k_{Z^2}, g_{Z^2}]$ or $Y_{V,i} \in (-k_V, g_V]$, where $-k_{Z2} < g_{Z2} \le h_{Z2}$ and $-k_V < g_V \le h_V$. Since it is difficult to obtain the joint properties of (4.2) and (4.3), the performances of this multivariate CUSUM scheme can be evaluated by simulation when the parameters of the process are on-target or changed.

For simultaneous use of (TM_i, TV_i) given in (2.3) and (2.4), the multivariate CUSUM control scheme for μ at the ith sample is

$$Y_{TM,i} = \max\{Y_{TM,i-1}, 0\} + (TM_i - k_{TM}) \tag{4.4}$$

and for Σ at the ith sample is

$$Y_{TV,i} = \max\{Y_{TV,i-1}, 0\} + (TV_i - k_{TV}), \tag{4.5}$$

where $Y_{TM,0} = w_{TM} \cdot I_{(w_{TM} \ge 0)}$, $Y_{TV,0} = w_{TV} \cdot I_{(w_{TV} \ge 0)}$, $k_{TM} \ge 0$ and $k_{TV} \ge 0$. This chart signals whenever $Y_{TM,i} \ge h_{TM}$ or $Y_{TV,i} \ge h_{TV}$.

For the VSI scheme, suppose that the sampling interval ;

d_1 is used when $Y_{TM,i} \in (g_{TM}, h_{TM}]$ or $Y_{TV,i} \in (g_{TV}, h_{TV}]$,

d_2 is used when $Y_{TM,i} \in (-k_{TM}, g_{TM}]$ or $Y_{TV,i} \in (-k_{TV}, g_{TV}]$,

where $-k_{TM} < g_{TM} \le h_{TM}$ and $-k_{TM} < g_{TV} \le h_{TV}$. Since it is difficult to obtain the joint properties of (4.4) and (4.5), the performances of this scheme can be evaluated by simulation when the parameters of the process are on-target or changed. The parameters h_D and g_D can be obtained by Markov chain or integral equation approach, and the parameters g_{Z2}, h_{Z2}, g_V, h_V, g_{TM}, h_{TM}, g_{TV} and h_{TV} can be obtained by simulations.

Brook and Evans(1972) developed a Markov chain approach for a univariate one-sided FSI CUSUM chart. In this paper, a modification of their model is considered for VSI multivariate CUSUM chart. Since the multivariate CUSUM statistic is continuous, the continuous state space of the statistic is partitioned into a finite number of discrete intervals and the probability distribution of the CUSUM is discretized.

Let the interval $(-\infty, \infty)$ be divided into in-control region $C = (-\infty, h]$ and out-of-control region $C' = (h, \infty)$. Suppose that the region C for the CUSUM statistic Y_i is partitioned into r states $E_1, E_2, \cdots, E_r$ where each interval corresponds to a state of Markov chain and region C' for the CUSUM value Y_i greater than h is an absorbing state. Since Y_i is continuous, let a discretized version $\tilde{Y}_i$ of $Y_i \in E_i$ be the midpoint of E_i. Suppose that the multivariate CUSUM chart signals when $Y_i \ge h$, the sampling interval d_1 is used when $Y_i \in (g, h]$ and the sampling interval d_2 is used when $Y_i \in [-k, g)$.

Then the transition probability matrix $P = [p_{ij}]$ can be partitioned as

$$P = \begin{bmatrix} Q & (I-Q)\underline{1} \\ \underline{0}' & 1 \end{bmatrix},$$

where Q is the $r \times r$ transition matrix corresponding to the transient state, $\underline{0}$ is an $r \times 1$ vector of 0's and $\underline{1}$ is the $r \times 1$ vector of 1's. The probability of moving from any state i to any other state j can be denoted as $p_{ij}(k) = P(Y_{k+1} \in E_j \mid Y_k \in E_i)$ for $i, j = 1, 2, \cdots, r+1$ and $k = 0, 1, 2, \cdots$. In this paper, $p_{ij}(k)$ will be written briefly as p_{ij}. Suppose that states $1, 2, \cdots, m$ used d_2 and states $m+1, m+2, \cdots, r$ used d_1.

Consider first the case $g > 0$. Then the state 1 corresponds to $Y_i \leq 0$ and $\tilde{Y}_i = 0$. Let $w = g/(m-1)$. Then for $j = 2, 3, \cdots, m$, state j corresonds to $(j-2)w < Y_i \leq (j-1)w$ and $\tilde{Y}_i = (j-3/2)w$. Let $v = (h-g)/(r-m)$. Then for $j = m+1, \cdots, r$, state j corresponds to $g+(j-m-1)v < Y_i \leq g+(j-m)v$ and $\tilde{Y}_i = g+(j-m-\frac{1}{2})v$.

The transition probabilities p_{ij} for Q are as follows : For $i = 1$,

$$p_{1j} = \begin{cases} P[z \leq 0] & j = 1 \\ P[(j-2)w < z \leq (j-1)w] & j = 2, \cdots, m \\ P[g+(j-m-1)v < z \leq g+(j-m)v] & j = m+1, \cdots, r. \end{cases}$$

For $i = 2, 3, \cdots, m$,

$$p_{ij} = \begin{cases} P[z \leq -(i-\frac{3}{2})w] & j = 1 \\ P[(j-i-\frac{1}{2})w < z \leq (j-i+\frac{1}{2})w] & j = 2, 3, \cdots, m \\ P[(m-i+\frac{1}{2})w+(j-m-1)v < z & \\ \quad \leq (m-i+\frac{1}{2})w+(j-m)v] & j = m+1, m+2, \cdots, r. \end{cases}$$

For $i = m+1, m+2, \cdots, r$,

$$p_{ij} = \begin{cases} P[z < -g-(i-m-\frac{1}{2})v] & j = 1 \\ P[-(m-j+1)w-(i-m-\frac{1}{2})v < z & \\ \quad \leq -(m-j)w-(i-m-\frac{1}{2})v] & j = 2, 3, \cdots, m \\ P[(j-i-\frac{1}{2})v < z \leq (j-i+\frac{1}{2})v] & j = m+1, \cdots, r. \end{cases}$$

For the case $g = 0$, only one state corresponding to $Y_i \le 0$ is needed to d_2 and thus $m = 1$. Let $w = h/(r-1)$. Then for $i = 1$,

$$p_{1j} = \begin{cases} P[z \le 0] & j = 1 \\ P[(j-2)w < z \le (j-1)w] & j = 2, \cdots, r. \end{cases}$$

For $i = 2, 3, \cdots, r$,

$$p_{ij} = \begin{cases} P[z \le -(i-\frac{3}{2})w] & j = 1 \\ P[(j-i-\frac{1}{2})w < z \le (j-i+\frac{1}{2})w] & j = 2, 3, \cdots, r. \end{cases}$$

For the case $g < 0$, two states are needed for nonpositive values Y_i. State 1 corresponds to $Y_i \le g$ where d_2 is used, and state 2 corresponds to $g < Y_i \le 0$ where d_1 is used. Thus $m = 2$. Let $w = h/(r-2)$. Then for state 1

$$p_{1j} = \begin{cases} P[z \le g] & j = 1 \\ P[g < z \le 0] & j = 2 \\ P[(j-3)w < z \le (j-2)w] & j = 3, \cdots, r. \end{cases}$$

For state 2, $p_{2j} = p_{1j}$ for $j = 1, 2, \cdots, r$.
For $i = 3, \cdots, r$,

$$p_{ij} = \begin{cases} P[z \le g-(i-\frac{5}{2})w] & j = 1 \\ P[g-(i-\frac{5}{2})w < z \le -(i-\frac{5}{2})w] & j = 2 \\ P[(j-i-\frac{1}{2})w < z \le (j-i+\frac{1}{2})w] & j = 3, 4, \cdots, r. \end{cases}$$

From the transition matrix P, we can obstain the fundamental matrix M as

$$M = (I-Q)^{-1} = [m_{ij}],$$

where m_{ij} is the expected number of visits to the transient state j before absorption, given that the Markov chain starts in transient state i. Let $\underline{b} = (b_1, \cdots, b_r)'$ where b_i is the sampling interval when $Y_i \in E_i$, $\underline{N} = (N_1, \cdots, N_r)'$ where N_j is NSS when the Markov chain starts in state j, $\underline{T} = (T_1, \cdots, T_r)'$ where T_j is TS when the Markov chain starts in state j.

Then, the ANSS vector is

$$E(\underline{N}) = M\underline{1}$$

and the variance vector of the NSS is

$$Var(\underline{N}) = (2M - I)\cdot E(\underline{N}) - (E(\underline{N}))^{(2)},$$

where $(E(\underline{N}))^{(2)}$ is a vector whose ith component is the square of the ith component of $E(\underline{N})$. Following Reynolds(1988), the ATS vector is

$$E(\underline{T}) = M\underline{b}$$

and the variance vector of $\underline{T}$ is

$$Var(\underline{T}) = MBE(2M-I) - (M\underline{b})^{(2)},$$

where B is a diagonal matrix with elements $b_1, b_2, \cdots, b_r$ and $(M\underline{b})^{(2)}$ is a vector whose ith component is the square of the ith component of $M\underline{b}$.

Let ATS(r) be ATS calculated using r states. Lucas and Crosier(1982) showed that an approximation of the continuous state ATS with a second degree polynomial in $1/r^2$ is a good approximation. This polynomial is of the form

$$ATS(r) = A + B/r^2 + C/r^4,$$

where A, B and C are the coefficients. For large r, the approximation A is called the asymptotic ATS. For VSI charts this approach can be used to estimate the ATS by taking ATS(r).

5. Computational Results and Conclusions

In order to evaluate the performances and compare the matched FSI and VSI multivariate Shewhart and CUSUM charts, some kinds of standards for comparison are necessary. For convenience, we let that the sampling interval of unit time $d = 1$ in FSI charts and two sampling intervals used as $d_1 = 0.1$ and $d_2 = 1.9$ for the VSI charts. In our computation, the ATS and ANSS of the chart when the process is in-control were fixed to be 200 and the sample size for each characteristic was five for $p = 4$ and 6. The types of shifts in the parameters when the process is out-of-control for comparison can be stated as follows :

Types of Shifts in Means

(M_1) one mean shifted with scale $\tau^2 = 1.0^2$.
(M_2) one mean shifted with scale $\tau^2 = 2.0^2$.
(M_3) one mean shifted with scale $\tau^2 = 3.0^2$.

Types of Shifts in Variances

(V_1) one variance is increased to $(1.1)^2$.
(V_2) one variance is increased to $(1.2)^2$.
(V_3) half of variance are increased to $(1.2)^2$.

Types of Shifts in Means and Variances

Types of (M_1, V_2) and (M_3, V_3) were considered.

After the reference values of all the proposed multivariate CUSUM schemes in this paper have been determined, the parameters h and g based on D_i were calculated by Markov chains with $r=100$. And the parameters h and g based on (Z_i^2, V_i) and TV_i for CUSUM scheme, and the ATS and ANSS for all the proposed types of shifts for Shewhart and CUSUM charts were obtained by simulation with 10,000 iterations. The performances for matched FSI and VSI multivariate Shewhart and CUSUM charts are given in Table 1 through 4. When small or moderate changes in the process have occurred, multivariate CUSUM charts are more efficient than Shewhart charts in our computation. When shifts for means and variances in the process have occurred, the multivariate CUSUM chart based on (Z_i^2, V_i) will be recommended. Numerical results for various reference values show that large reference values are efficient for large shifts and smaller reference values are efficient for small shifts in multivariate CUSUM charts. For all the proposed multivariate control charts and all kinds of shifts, VSI scheme is more efficient than FSI scheme.

〈 Table 1 〉 ATS for matched FSI and VSI multivariate Shewhart charts (p = 4)

Types of Shift	D_i FSI	D_i VSI	(Z_i^2, V_i) FSI	(Z_i^2, V_i) VSI	(TM_i, TV_i) FSI	(TM_i, TV_i) VSI
no shift	200.0	200.0	200.0	200.0	200.0	200.0
M_1	116.9	104.6	85.0	72.7	84.0	72.1
M_2	34.2	21.9	15.3	8.3	14.7	8.2
M_3	9.1	3.8	3.7	1.7	3.7	1.7
V_1	106.9	95.4	118.3	107.4	155.7	150.3
V_2	58.4	46.1	67.3	54.7	111.3	102.4
V_3	25.8	16.3	32.0	21.2	72.3	61.4
(M_1, V_2)	37.6	26.9	34.2	24.5	42.1	34.0
(M_3, V_3)	4.1	1.8	3.0	1.5	3.1	1.6

〈 Table 2 〉 ATS for matched FSI and VSI multivariate CUSUM charts (p = 4)

Types of Shift	based on D_i FSI	based on D_i VSI	based on (Z_i^2, V_i) FSI	based on (Z_i^2, V_i) VSI	based on (TM_i, TV_i) FSI	based on (TM_i, TV_i) VSI
no shift	200.0	200.0	200.0	200.0	200.0	200.0
M_1	66.4	49.5	39.0	28.0	38.8	27.9
M_2	17.5	10.5	8.5	5.4	8.4	5.3
M_3	7.8	4.8	3.8	2.3	3.8	2.3
V_1	63.0	45.8	72.6	54.1	124.5	111.3
V_2	31.3	19.1	37.2	24.2	74.7	58.6
V_3	15.6	8.6	19.3	11.7	40.5	28.0
(M_1, V_2)	21.5	12.4	21.7	13.2	24.7	17.0
(M_3, V_3)	5.4	2.7	3.6	2.2	3.6	2.3
	h_D = 59.1546		h_{Z^2} = 25.9792	h_V = 65.9892	h_{TM} = 25.8564	h_{TV} = 87.6612
	g_D = 11.1494		g_{Z^2} = 6.2113	g_V = 21.2136	g_{TM} = 6.1549	g_{TV} = 28.1912
	k_D = 20.5		k_{Z^2} = 4.5	h_V = 16.5	k_{TM} = 4.5	k_{TV} = 16.0

〈 Table 3 〉 ATS for matched FSI and VSI multivariate Shewhart charts (p = 6)

Types of Shift	D_i FSI	D_i VSI	(Z_i^2, V_i) FSI	(Z_i^2, V_i) VSI	(TM_i, TV_i) FSI	(TM_i, TV_i) VSI
no shift	200.0	200.0	200.0	200.0	200.0	200.0
M_1	129.9	118.6	100.6	88.5	92.1	79.5
M_2	45.1	31.0	20.6	12.2	18.9	10.6
M_3	13.0	5.9	4.8	2.1	4.5	2.0
V_1	122.1	111.1	131.9	122.0	137.1	129.5
V_2	70.7	58.1	80.4	67.9	87.5	77.8
V_3	19.2	10.8	24.8	14.8	28.4	20.7
(M_1, V_2)	48.6	36.6	44.4	33.4	42.6	33.6
(M_3, V_3)	4.2	1.9	3.3	1.6	3.1	1.6

〈 Table 4 〉 ATS for matched FSI and VSI multivariate CUSUM charts (p = 6)

Types of Shift	based on D_i FSI	based on D_i VSI	based on (Z_i^2, V_i) FSI	based on (Z_i^2, V_i) VSI	based on (TM_i, TV_i) FSI	based on (TM_i, TV_i) VSI
no shift	200.0	200.0	200.0	200.0	200.0	200.0
M_1	77.7	60.3	48.0	36.2	46.4	34.3
M_2	21.9	13.5	10.6	7.1	10.5	6.9
M_3	9.8	6.1	4.8	3.0	4.7	2.8
V_1	74.2	56.0	84.5	65.8	124.5	111.3
V_2	38.4	24.5	46.2	31.6	72.8	55.7
V_3	13.1	7.2	16.6	10.4	27.9	17.8
(M_1, V_2)	26.8	16.1	27.3	17.7	29.7	20.3
(M_3, V_3)	5.8	2.9	4.2	2.7	4.2	2.1
	$h_D = 76.1056$		$h_{Z^2} = 34.0072$	$h_V = 85.7303$	$h_{TM} = 33.4676$	$h_{TV} = 167.3137$
	$g_D = 15.4204$		$g_{Z^2} = 9.0490$	$g_V = 29.0503$	$g_{TM} = 8.5244$	$g_{TV} = 56.6585$
	$k_D = 30.5$		$k_{Z^2} = 6.5$	$h_V = 24.5$	$k_{TM} = 6.5$	$k_{TV} = 115.0$

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