A Modified Test for the Hollander and Proschan's Test Against Decreasing Mean Residual Life Alternatives

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Abstract

In this paper we develop a test for alternatives representing decreasing mean residual life. The test statistic for decreasing mean residual life, K_{1n}, is a modified version of Hollander and Proschan's test V^* and critical constants and large sample approximation are shown to make the test readily applicable. Consistency is also shown for the tests based on K_{1n}. And small sample powers for four alernatives are obtained.

Key Words : exponential distribution, DMRL distribution, reliability, test for exponentiality, survival function.

1. Introduction

Statisticians have found it useful to categorize failure distributions, F such that $F(t)=0$ for $t < 0$, according to monotonicity properties of the failure rate, the average failure rate and the mean residual life. These categories are useful for modeling situations where items improve or deteriorate with age.

Let T be a nonnegative continuous random variable denoting the life time of a system and have an absolutely continuous distribution $F(t)$, survival function $\overline{F}(t)$, probability density function (p.d.f.) $f(t)$ and finite mean μ_F, i.e., $E_F[T] = \mu_F$.

The mean residual life (MRL) function of T, given that t units of time elapsed, is defined as;

$$\begin{aligned} m_F(t) &= E_F[T-t \mid T>t], \qquad t \geq 0 \\ &= \frac{1}{\overline{F}(t)} \int_t^\infty \overline{F}(u)du, \quad t \geq 0 . \end{aligned}$$

As the name implies, $m_F(t)$ is the mean value of a system's remaining life, after it has survived up to time t. In some cases, $m_F(t)$ could be called mean residual repair time. Also, the knowledge of the MRL function completely determines the survival function as follows:

$$\bar{F}(t) = \frac{m_F(0)}{m_F(t)} \exp\{ -\int_0^t \frac{1}{m_F(u)} du \}.$$

That is, there is a one to one correspondence between survival function and MRL function. The MRL is constant, i.e. $m_F(t) = \mu_F$, if and only if the distribution is exponential. As alternatives to the exponential distribution, Bryson and Siddiqui (1969) introduced the following classes by using the MRL function.

Definition A life distribution F is said to be decreasing mean residual life (DMRL) if $m_F(t)$ is decreasing in t. Similarly, F is increasing mean residual life (IMRL) if $m_F(t)$ is increasing in t.

The class of DMRL distribution is useful in modelling situations where life lengths of systems deteriorate with age. For the DMRL system, the older it is, the shorter the remaining life on the average is.

2. The Proposed Test Statistic

Consider the problem of testing

$H_0 : F(t) = 1-\exp(-\lambda t)$ $(t \geq 0, \lambda > 0)$ for λ unspecified (2.1)

versus

$H_1 : F(t)$ is in the DMRL class but not exponential (2.2)

$(H_1' : F(t)$ is in the IMRL class but not exponential) (2.3)

on the basis of a random variable from the failure distribution F.

Recall Hollander and Proschan's the DMRL test statistic (1975). They proposed the test statistic as follows;

$$V^* = \frac{\widehat{\Delta(F)}}{\hat{\mu}_F} = \frac{V}{\bar{X}_n}$$

where

$$\Delta(F) = \int_0^\infty \int_t^\infty \bar{F}(t)\bar{F}(s) \{ m_F(t) - m_F(s) \} dF(s) dF(t)$$

and

$$V = n^{-4} \sum_{i=1}^{n} \left(\frac{4}{3} i^3 - 4ni^2 + 3n^2 i - \frac{1}{2} n^3 + \frac{1}{2} n^2 - \frac{1}{2} i^2 + \frac{1}{6} i \right) X_i .$$

They derived the estimates μ_F and $\Delta(F)$ respectively and constructed the test statistic.

Consider the following parameter

$$T_1(F) = \frac{\Delta(F)}{\mu_F} .$$

Intuitively, $T_1(F) > 0$ favours $H_1(T_1(F) < 0$ favours $H_1')$. Let the renewal or equilibrium distribution corresponding to $F(t)$ be $G(t)$. Then it is well known that

$$\begin{aligned} \bar{G}(t) &= 1 - G(t) \\ &= \frac{1}{\mu_F} \int_t^\infty \bar{F}(u) du. \end{aligned}$$

Then we can rewrite $T_1(F)$ as follows:

$$\begin{aligned} T_1(F) &= \int_0^\infty \int_t^\infty \{ \bar{F}(s)\bar{G}(t) - \bar{F}(s)\bar{G}(t) \} dF(s)dF(t). \\ &= \int_0^\infty \left(\frac{1}{3} F^3(t) - F^2(t) + \frac{1}{2} F(t)\right) dG(t). \end{aligned}$$

For life time data the random variable $X_1 < X_2 < \ldots < X_n$ are naturally ordered. The empirical distribution is $F_n(X_i) = \frac{i}{n}$, $i = 0, 1, \ldots, n$ (where $X_0 = 0$). The empirical survival function is $\bar{F}_n(X_i) = \frac{n-i}{n}$, $i = 0, 1, \ldots, n$. Let A_F be the area under $\bar{F}_n(x)$, and A'_F be the area under $\bar{F}_n(X)$ after X_i, then:

$$\begin{aligned} A_F &= \sum_{j=0}^{n-1} \frac{n-j}{n} (X_{j+1} - X_j) = \bar{X}_n \\ A'_F &= \sum_{j=i}^{n-1} \frac{n-j}{n} (X_{j+1} - X_j) \\ &= \frac{1}{n} \sum_{j=i+1}^{n} D_j \end{aligned}$$

where $D_j = (n-j+1)(X_j - X_{j-1}), j = 1, 2, \ldots, n$. Putting $i = 0$ in the expression for

A^i_F, we get $A^0_F = A_F = \overline{X}_n$; thus $\overline{X}_n = \frac{1}{n}\sum_{j=1}^n D_j$. Let $T_n = n\overline{X}_n$. Now,

$$\overline{G}_n(X_i) = \frac{A^i_F}{A_F} = \frac{\sum_{j=i+1}^n D_j}{\sum_{j=1}^n D_j}, \quad i=0, 1, ..., n-1; \quad \overline{G}_n(X_n)=0.$$

Therefore we can obtain

$$\begin{aligned}\Delta G_n(X_i) &= -(\overline{G}_n(X_i)-\overline{G}_n(X_i))\\ &= \frac{D_j}{\sum_{j=1}^n D_j}, \quad i=1, 2, ..., n.\end{aligned}$$

The sample analogue of $T_1(F)$ will be taken as the test statistic; thus the test statistic is

$$K_{1n} = \sum_{i=1}^n w_{1i} D_i / T_n,$$

where $w_{1i} = \frac{1}{3}(\frac{i}{n})^3 - (\frac{i}{n})^2 + \frac{1}{2}(\frac{i}{n})$. If F is in the DMRL (IMRL) class, but not exponential, we expect K_{1n} to be positive (negative).

Since K_{1n} is scale invariant under H_0, we assume as in Langenberg and Srinivasan (1979), that $\lambda = \frac{1}{2}$ where λ is the parameter in negative exponential $f(t, \lambda) = \lambda\exp(-\lambda t)$; then the D_i's are independent χ^2_2 variates and Theorem 2.4 of Box (1954) gives

$$\begin{aligned}Pr(K_{1n} > k) &= Pr(\sum_{i=1}^n w_{1i} D_i / \sum_{i=1}^n D_i > k)\\ &= Pr(\sum_{i=1}^n (w_{1i} - k) D_i > 0)\\ &= \sum_{i=1}^n \prod_{i\neq j=1}^n \{\frac{w_{1i}-k}{w_{1i}-w_{1j}}\}\delta_i,\end{aligned}$$

where $\delta_i = 1$ for $w_{1i} > k$ and 0 otherwise.

For testing H_0 versus H_1, we need to tabulate critical values k satisfying $P_0(K_{1n} < k) = \alpha$ or $P_0(K_{1n} > k) = \alpha$ for various values of test size α and sample size n. Table 1 gives the critical values of $(210n)^{1/2}K_{1n}$ for α = 0.10, 0.05, 0.01, with sample sizes n = 5(1)30 and 35. (P_0 denotes the probability under H_0).

3. Properties of the Proposed Test Statistic

3.1 The Asymptotic Distribution When H_0 is True

We shall now prove that the proposed tests are asymptotically nomally distributed under rather general assumptions on F_0.

The test statistic K_{1n} can be written as

$$K_{1n} = \frac{1}{\bar{X}_n} \frac{1}{n} \sum_{i=1}^{n} (w_{1i}' + p(\frac{i^a}{n^b}))X_i$$
$$\simeq \frac{1}{\bar{X}_n} \frac{1}{n} \sum_{i=1}^{n} w_{1i}' X_i \quad (3.1)$$

where $p(\frac{i^a}{n^b})$ denotes a sum of terms of the form $\frac{i^a}{n^b}$ with $a < b$ and

$$w_{1i}' = \frac{4}{3}(\frac{i}{n})^3 - 4(\frac{i}{n})^2 + 3(\frac{i}{n}) - \frac{1}{2}. \quad (3.2)$$

These terms can be ignored without disturbing the asymptotic properties of K_{1n}. From (3.1) and (3.2),

$$K_{1n} \simeq \frac{K_{1n}^*}{\bar{X}_n}$$
$$= \frac{1}{\bar{X}_n} \frac{1}{n} \sum_{i=1}^{n} J_1(\frac{i}{n}) X_i,$$

where $K_{1n}^* = \frac{1}{n} \sum_{i=1}^{n} w_{1i}' X_i$ and $J_1(u) = \frac{4}{3} u^3 - 4u^2 + 3u - \frac{1}{2}$.

Theorem Let F be a life distribution and

$$\mu(J_1, F) = \int_0^\infty x J_1 (F(x)) dF(x) \quad (3.3)$$

and

$$\sigma^2(J_1, F) = \int_0^\infty \int_0^\infty J_1(F(x)) J_1(F(y)) [F(\min(x,y)) - F(x)F(y)] dx dy \quad (3.4)$$

and assume that $\int_0^\infty x^2 dF(x) < \infty$, $\sigma^2(J_1, F) > 0$; then

$$\sqrt{n}\,(K_{1n}^{*}-\mu(J_1, F)) \xrightarrow{d} N(0, \sigma^2(J_1, F)).$$

Proof. Since $J_1(u)$ is bounded and continuous on the unit interval, this theorem can be proved by using Theorem 2 and 3 of Stigler (1974).

From the result for K_{1n}^{*} and Slusky's theorem (Cramér, 1946, pp. 254−55), the limiting distribution of $\sqrt{n}\,(K_{1n} - \frac{\mu(J_1, F)}{\mu_F})$ is $N(0, \frac{\sigma^2(J_1, F)}{\mu_F^2})$

When doing calculations under H_0 we can, since K_{1n} is scale invariant, take F to be exponential with scale parameter $\lambda = 1$. For $F_0(t)=1-\exp(-t)$, we get $\mu_F = 1$, $\mu(J_1, F_0)=0$ and $\frac{\sigma^2(J_1, F)}{\mu_F^2} = \frac{1}{210}$. Thus, under H_0,

$\sqrt{210n}\,K_{1n}$ is asymptotically $N(0, 1)$.

The DMRL test procedure rejects H_0 in favor of H_1 : F is DMRL at the approximate α level if

$$\sqrt{210n}\,K_{1n} \geq z_\alpha,$$

where z_α is the upper α−quantile of the standard normal distribution Analoguously, the approximate α level test of H_0 versus H_1': F is IMRL rejects H_0 if

$$\sqrt{210n}\,K_{1n} \leq -z_\alpha.$$

3.2 Consistency of the Proposed Statistic

We shall show consistency of the test by establishing that the power tends to one when $n \rightarrow \infty$.

Assume that F is in the DMRL class and is continuous and $\int_0^\infty t^2 dF(t) < \infty$ and $\sigma^2(J_1, F) > 0$, then the power function of the asymptotic test is

$$\prod(T_1(F)) = P(K_{1n} > \frac{1}{\sqrt{210n}} z_\alpha),$$

where z_α is the upper α−quantile of the standard normal distribution.

$$\prod(T_1(F)) = \Phi\left(\frac{\frac{-1}{\sqrt{210n}} z_\alpha + \frac{\mu(J_1, F)}{\mu_F}}{\frac{\sigma^2(J_1, F)}{\sqrt{n}\,\mu_F}} \right).$$

From the results given in Chap.2,

$$T_1(F) = \frac{1}{\mu_F}\int_0^\infty \left(-\frac{1}{3}\bar{F}^4(t) + \frac{1}{2}\bar{F}^3(t) - \frac{1}{6}\bar{F}(t)\right)dt$$

$$\mu(J_1, F) = \int_0^\infty \left(-\frac{1}{3}\bar{F}^4 + \frac{1}{2}\bar{F}^3(t) - \frac{1}{6}\bar{F}(t)\right)dt.$$

Therefore

$$\frac{\mu(J_1, F)}{\mu_F} = T_1(F).$$

Thus

$$\prod(T_1(F)) = \Phi\left(\frac{-\frac{1}{\sqrt{210n}} z_\alpha + T_1(F)}{\frac{\sigma(J_1, F)}{\sqrt{n}\,\mu_F}} \right)$$

When F is in the DMRL class then $T_1(F) > 0$ and

$\prod(T_1(F)) \to \Phi(+\infty) = 1$, as $n \to \infty$.

4. Small Sample Power and Conclusion

A Monte Carlo study was carried out to estimate the power of the K_{1n} test for the following four alteratives considered corresponding to significance levels α = 0.05 and 0.1.

(a) Gamma distribution

$$G_\theta(x) = \int_0^x \frac{1}{\Gamma(1+\theta)} \exp(-t)t^\theta dt, \quad x \geq 0, \quad \theta \geq 0.$$

(b) Weibull distribution

$$W_\theta(x) = 1 - \exp(-x^{1+\theta}), \quad x \geq 0, \quad \theta \geq 0.$$

(c) Linear Failure Rate distribution

$$L_\theta(x) = 1 - \exp(-(x + \theta x^2/2)), \quad x \geq 0, \quad \theta \geq 0.$$

(d) Makeham distribution

$$M_\theta(x) = 1 - \exp(-(x + \theta(x + e^{-x} - 1))), \quad x \geq 0, \quad \theta \geq 0.$$

We set the sample size n = 2, 3, ... 30.

Tables 2 through 5 show a comparative study of small sample powers for two tests for DMRL. The tests selected are Hollander-Proschan's (1975) V^* test for DMRL [HP] and the K_{1n} test [KK]. The tables show that the K_{1n} test is, clearly, superior to V^* test on most occasions.

〈 Table 1 〉 Critical values of the decreasing mean residual life statistic $\sqrt{210n}\,K_{1n}$

	Lower tail			Upper tail		
n	$\alpha=0.01$	$\alpha=0.05$	$\alpha=0.1$	$\alpha=0.1$	$\alpha=0.05$	$\alpha=0.01$
2	−3.3730	−3.2022	−2.9887	0.4269	0.6404	0.8112
3	−3.7058	−3.1155	−2.6732	0.6253	0.9343	1.3599
4	−3.7054	−2.9076	−2.4066	0.7329	1.0105	1.4081
5	−3.6212	−2.7398	−2.2364	0.7860	1.0687	1.4816
6	−3.5273	−2.6201	−2.1276	0.8283	1.1092	1.5309
7	−3.4433	−2.5318	−2.0425	0.8581	1.1407	1.5773
8	−3.3726	−2.4617	−1.9773	0.8823	1.1673	1.6151
9	−3.3127	−2.4043	−1.9249	0.9024	1.1876	1.6479
10	−3.2610	−2.3565	−1.8819	0.9124	1.2089	1.6761
11	−3.3159	−2.3160	−1.8457	0.9341	1.2257	1.7009
12	−3.1762	−2.2811	−1.8149	0.9470	1.2406	1.7228
13	−3.1409	−2.2508	−1.7883	0.9584	1.2539	1.7424
14	−3.1094	−2.2241	−1.7650	0.9687	1.2685	1.7601
15	−3.0810	−2.2004	−1.7444	0.9779	1.2766	1.7762
16	−3.0552	−2.1792	−1.7260	0.9864	1.2865	1.7909
17	−3.0317	−2.1600	−1.7094	0.9942	1.2957	1.8046
18	−3.0103	−2.1426	−1.6945	1.0013	1.3041	1.8171
19	−2.9905	−2.1268	−1.6310	1.0079	1.3119	1.3237
20	−2.9723	−2.1123	−1.6686	1.0140	1.3191	1.8394
21	−2.9555	−2.0990	−1.6572	1.0196	1.3259	1.3495
22	−2.9398	−2.0867	−1.6468	1.0249	1.3322	1.8590
23	−2.9251	−2.0752	−1.6371	1.0299	1.3381	1.8678
24	−2.9115	−2.0646	−1.6280	1.0346	1.3437	1.8762
25	−2.8986	−2.0547	−1.6197	1.0390	1.3490	1.8841
26	−2.8866	−2.0454	−1.6118	1.0432	1.3540	1.8916
27	−2.8752	−2.0367	−1.6044	1.0471	1.3587	1.8987
28	−2.8644	−2.0284	−1.5975	1.0509	1.3632	1.9054
29	−2.8543	−2.0207	−1.5910	1.0544	1.3675	1.9118
30	−2.8446	−2.0134	−1.5849	1.0578	1.3716	1.9180
35	−2.8030	−1.9820	−1.5586	1.0727	1.3895	1.9448

〈 Table 2 〉 Comparison of small powers

$\alpha = 0.05$ and $\theta = 1$

	Gamma		Weibull		L. F. R.		Makeham	
n	HP	KK	HP	KK	HP	KK	HP	KK
2	.0743	.0748	.1101	.1011	.0612	.0614	.0579	.0579
3	.1086	.1118	.1878	.1939	.0777	.0793	.0684	.0708
4	.1209	.1318	.2256	.2480	.0859	.0904	.0737	.0739
5	.1259	.1362	.2583	.2878	.0894	.0921	.0711	.0739
6	.1328	.1499	.2791	.3204	.1015	.1056	.0815	.0837
7	.1380	.1554	.3049	.3484	.1060	.1100	.0818	.0824
8	.1365	.1561	.3276	.3796	.1208	.1237	.0878	.0888
9	.1451	.1642	.3502	.4029	.1247	.1276	.0886	.0916
10	.1489	.1660	.3799	.4333	.1367	.1407	.0925	.0957
11	.1555	.1746	.3936	.4500	.1476	.1499	.0982	.1009
12	.1658	.1826	.4214	.4751	.1579	.1609	.1065	.1081
13	.1677	.1841	.4361	.4923	.1711	.1736	.1087	.1126
14	.1677	.1841	.4596	.5119	.1717	.1769	.1087	.1112
15	.1801	.1950	.4825	.5348	.1848	.1904	.1150	.1167
20	.1937	.2193	.5667	.6347	.2228	.2360	.1255	.1337
25	.2395	.2556	.6770	.7200	.2773	.2829	.1469	.1501
30	.2669	.2833	.7424	.7792	.3464	.3530	.1704	.1722

〈 Table 3 〉 Comparison of small powers

$\alpha = 0.1$ and $\theta = 1$

	Gamma		Weibull		L. F. R.		Makeham	
n	HP	KK	HP	KK	HP	KK	HP	KK
2	.1440	.1439	.1916	.1916	.1184	.1184	.1111	.1111
3	.2101	.2113	.3311	.3320	.1540	.1549	.1366	.1363
4	.2254	.2361	.3837	.4081	.1665	.1709	.1424	.1437
5	.2377	.2552	.4386	.4704	.1746	.1788	.1406	.1425
6	.2515	.2692	.4740	.5154	.1896	.1937	.1498	.1541
7	.2608	.2799	.5079	.5558	.1986	.2045	.1553	.1593
8	.2636	.2843	.5417	.5882	.2210	.2276	.1668	.1685
9	.2733	.2953	.5642	.6123	.2326	.2369	.1687	.1729
10	.2795	.3018	.5936	.6431	.2434	.2490	.1777	.1820
11	.2868	.3084	.6169	.6667	.2585	.2652	.1846	.1877
12	.2947	.3194	.6410	.6879	.2714	.2760	.1918	.1967
13	.3048	.3288	.6607	.7083	.2899	.2970	.2025	.2061
14	.3059	.3311	.6824	.7216	.2991	.3065	.2014	.2054
15	.3196	.3424	.7021	.7467	.3168	.3213	.2122	.2149
20	.3508	.3769	.7939	.8300	.3780	.3837	.2361	.2390
25	.4006	.4213	.8573	.8846	.4402	.4460	.2610	.2632
30	.4336	.4529	.9023	.9221	.5208	.5242	.2906	.2942

〈 Table 4 〉 Comparison of small powers

$\alpha = 0.05$ and $\theta = 2$

	Gamma		Weibull		L. F. R.		Makeham	
n	HP	.KK	HP	KK	HP	KK	HP	KK
2	.0930	0935	.1452	.1454	.0655	.0658	.0610	.0611
3	.1614	.1663	.3484	.3625	.0899	.0931	.0787	.0807
4	.1914	.2108	.4660	.5122	.1049	.1083	.0872	.0909
5	.2104	.2364	.5343	.6018	.1105	.1161	.0881	.0905
6	.2121	.2457	.5844	.6685	.1243	.1299	.0981	.1024
7	.2241	.2626	.6165	.7180	.1339	.1434	.1023	.1059
8	.2284	.2722	.6428	.7484	.1528	.1600	.1135	.1175
9	.2352	.2790	.6615	.7692	.1604	.1692	.1157	.1195
10	.2375	.2781	.6831	.7876	.1754	.1850	.1262	.1306
11	.2517	.2941	.7029	.8043	.1926	.2001	.1326	.1373
12	.2548	.2999	.7234	.8178	.2049	.2138	.1463	.1500
13	.2688	.3131	.7369	.8363	.2245	.2332	.1534	.1562
14	.2725	.3155	.7556	.8488	.2332	.2413	.1529	.1568
15	.2792	.3252	.7683	.8577	.2485	.2573	.1624	.1688
20	.3110	.3669	.8308	.9092	.3065	.3260	.1894	.2013
25	.3756	.4140	.8959	.9425	.3884	.3994	.2264	.2309
30	.4256	.4661	.9318	.9620	.4627	.4729	.2660	.2730

〈 Table 5 〉 Comparison of small powers

$\alpha = 0.1$ and $\theta = 2$

	Gamma		Weibull		L. F. R.		Makeham	
n	HP	KK	HP	KK	HP	KK	HP	KK
2	.1945	.1945	.2850	.2849	.1285	.1283	.1195	.1191
3	.2899	.2924	.5571	.5664	.1737	.1762	.1548	.1564
4	.3279	.3520	.6764	.7112	.1963	.2008	.1667	.1708
5	.3591	.3933	.7560	.8018	.2139	.2200	.1716	.1782
6	.3734	.4155	.8049	.8545	.2308	.2376	.1855	.1895
7	.3934	.4381	.8371	.8858	.2451	.2558	.1908	.1969
8	.4081	.4540	.8654	.9123	.2742	.2836	.2096	.2180
9	.4166	.4654	.8827	.9296	.2915	.3012	.2191	.2241
10	.4199	.4724	.8943	.9414	.3121	.3183	.2295	.2337
11	.4420	.4893	.9064	.9512	.3299	.3405	.2384	.2465
12	.4521	.5039	.9190	.9596	.3414	.3522	.2504	.2568
13	.4594	.5233	.9371	.9713	.3869	.3982	.2684	.2766
14	.4842	.5350	.9411	.9726	.4120	.4206	.2820	.2903
15	.2792	.3252	.7683	.8577	.2485	.2573	.1624	.1688
20	.5352	.5836	.9662	.9842	.4911	.5001	.3279	.3353
25	.5892	.6253	.9828	.9927	.5708	.5795	.3755	.3797
30	.6402	.6773	.9918	.9960	.6414	.6494	.4228	.4316

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