Some Lower Bound of Cramer-Rao type for Median-Unbiased Estimates

Beong-Soo So
Dept. of Statistics, Ewha Womans University

Abstract

We construct a new lower bound of Cramer-Rao type for the median-unbiased estimator in the presence a nuisance parameter. We also identify useful necessary and sufficient conditions for the attainability of the lower bound. Some applications including the analysis of censored reliability data are considered as examples.

1. Introduction

Let μ be a Lebesque measure on the Euclidean space R^n. Let $P = \{ P_\theta : \theta \in \Theta \}$ be the family of distributions on R^n with desity functions $f(x ; \theta)$, $\theta \in \Theta$ with respect to measure μ. Here x denotes the vector $(x_1, \cdots, x_n)$ and θ denotes the unknown parameter with values in the parameter space Θ. We also denote X as a random vector $(X_1, \cdots, X_n)$ with distribution P_θ, $\theta \in \Theta$.

In this paper we are interested in estimating a real-valued function $g(\theta)$ of the parameter. One reasonable criterion of a good estimator which is widely used in the literature is the concept of the unbiasedness of the estimator. In this paper we introduce two most popular definitions of the unbiasedness as follows ;

Definition. An estimator $\delta(x)$ of $g(\theta)$ is called *mean-unbiased* if

$$E_\theta[\delta(X)] = g(\theta) \qquad \text{for all} \quad \theta \in \Theta . \tag{1.1}$$

Definition. An estimator $\delta(X)$ of $g(\theta)$ is said to be *median-unbiased* if

$$P_\theta[\delta(X) \le g(\theta)] = P_\theta[\delta(X) \ge g(\theta)] = \frac{1}{2} \qquad \text{for all } \theta \in \Theta . \tag{1.2}$$

We also define $f_\delta(y;\theta)$ as a density function of the estimator $\delta(X)$ in this paper.

If $\delta(X)$ is a mean-unbiased estimator of $g(\theta)$, there exists a well-known lower bound for the variance of the estimator named as Cramer-Rao lower bound in the literature ;

$$Var_\theta[\delta(X)] \geq \{g'(\theta)\}^2 / I_2(\theta) \qquad \text{for all} \quad \theta \in \Theta \tag{1.3}$$

where $I_2(\theta) = E_\theta[\partial \log f(X;\theta)/\partial\theta]^2$ is the Fisher infomation number. See Lehmann (1983) for more details.

On the other hand, several versions of the analogue of the Cramer-Rao lower bound for median-unbiased estimatiors were also proposed in the literature.

As a first step in this direction, Alamo (1964) introduced the quantity $f_\delta(g(\theta);\theta)$ as a reasonable measure of the degree of *concentration* of the estimator $\delta(X)$ around the estimand $g(\theta)$ and proposed its reciprocal quantity $1/f_\delta(g(\theta);\theta)$ as a new measure of *dispersion* of the estimator. Then he obtained following lower bound for median-unbiased estimator ;

$$\frac{1}{2f_\delta(g(\theta);\theta)} \geq |g'(\theta)| / I_2(\theta)^{\frac{1}{2}} \tag{1.4}$$

where $I_2(\theta)$ is the usual Fisher information number

$$I_2(\theta) = E[(\partial/\partial\theta)\log f(X;\theta)]^2.$$

Recently several sharper lower bounds for the left hand side of the above inequality were proposed by Sung, Stangenhaus and David (1990) and So (1993). For example Sung et al. (1990) obtained the improved result ;

$$\frac{1}{2f_\delta(g(\theta);\theta)} \geq |g'(\theta)| / I_1(\theta) \qquad \text{for all} \quad \theta \in \Theta \tag{1.5}$$

where $I_1(\theta)$ is an L_1-analogue of Fisher information

$$I_1(\theta) = E_\theta|(\partial/\partial\theta)\log f(X:\theta)|.$$

Most recent result in this direction is So (1993) who derived the best available bound ;

$$1/\{2f_\delta(g(\theta);\theta)\} \geq |g'(\theta)| / I_1^*(\theta) \qquad \text{for all} \quad \theta \in \Theta \tag{1.6}$$

where $I_1^*(\theta)$ is a modified L_1-Fisher information number

$$I_1^*(\theta) = E_\theta[\,|\,\partial \log f(X;\theta)/\partial\theta - k\,|\,] \tag{1.7}$$

and $k = median_\theta[\partial \log f(X;\theta)/\partial\theta]$. Note that the lower bound of (1.6) is strictly greater than that of (1.5) unless k happens to be zero. On the other hand, when $\theta=(\theta_1, \theta_2)$ and $g(\theta_1)$ is our parameter of interest in the presence of nuisance parameter θ_2, Sung (1993) improved the bound (1.5) and obtained following lower bound ;

$$\frac{1}{2f_\delta(g(\theta_1);\theta)} \geq |g'(\theta_1)|/I_3(\theta) \tag{1.8}$$

where $I_3(\theta) = E_\theta\,|(\partial/\partial\theta_1)\log f(X;\theta) - k^*(\partial/\partial\theta_2)\log f(X;\theta)|$ and

$$k^* = median_\theta\left[\frac{\partial \log f(X;\theta)}{\partial\theta_1} / \frac{\partial \log f(X;\theta)}{\partial\theta_2}\right].$$

In this paper we will try to improve the bound (1.8) by employing the simple centering technique which was crucially used in deriving the bound (1.6). Essentially we will establish the following result ;

$$\frac{1}{2f_\delta(g(\theta_1);\theta)} \geq |g'(\theta_1)|/I_1^*(\theta) \tag{1.9}$$

where $I_1^*(\theta) = \inf_{(k_1,k_2)\in R^2} E_\theta\,|(\partial \log f(X;\theta)/\partial\theta_1) - k_1(\partial \log f(X;\theta)/\partial\theta_2) - k_2\,|$.

Note again that lower bound of (1.9) is strictly greater than that of (1.8) except for the special case $k_2=0$.

We will also identify the simple necessary and sufficient conditions for the attainability of the lower bound (1.9) which provide useful tools for contructing optimal median-unbiased estimates. Finally some examples including the application to the analysis of censored reliability data are given as illustrations.

2. Lower Bound

Let Δ be a real number such that both (θ_1, θ_2) and $(\theta_1+\Delta, \theta_2)$ belong to the parameter space $\Theta \subset R^2$. Then, by the definition of median-unbiased estimator, we have the following identities ;

$$E_{(\theta_1, \theta_2)} [\mathrm{sgn}(\delta(X) - g(\theta_1)] = 0, \tag{2.1}$$

$$E_{(\theta_1+\Delta, \theta_2)} [\mathrm{sgn}(\delta(X) - g(\theta_1+\Delta))] = 0. \tag{2.2}$$

Subtracting (2.1) from (2.2) we get the identity ;

$$\int [f(x;\theta_1+\Delta;\theta_2) - f(x;\theta_1, \theta_2)] \mathrm{sgn}[\delta(x) - g(\theta_1+\Delta)] d\mu$$
$$+ E_{(\theta_1, \theta_2)} [\mathrm{sgn}[\delta(X) - g(\theta_1+\Delta)] - \mathrm{sgn}[\delta(X) - g(\theta_1)]] = 0. \tag{2.3}$$

Now we may write (2.1) and (2.2) in the form ;

$$\int \mathrm{sgn}(\delta(x) - g(\theta_1)) f(x;\theta_1, \theta_2) d\mu = 0,$$
$$\int \mathrm{sgn}(\delta(x) - g(\theta_1+\Delta)) f(x;\theta_1+\Delta, \theta_2) d\mu = 0. \tag{2.4}$$

Differentiating (2.4) with respect to θ_2 and allowing interchange of the integral and the derivative sign, we obtain the identity ;

$$\int \mathrm{sgn}[\delta(x) - g(\theta_1+\Delta)] \frac{\partial f(x;\theta_1+\Delta, \theta_2)}{\partial \theta_2} d\mu = 0. \tag{2.5}$$

Multiplying (2.5) by $k_1\Delta$ and subtracting it from (2.3), we have the following identity ;

$$2[F_\delta(g(\theta\ +\Delta);\theta_1, \theta_2) - F_\delta(g(\theta_1);\theta_1, \theta_2)]$$
$$= \int [f(x;\theta_1+\Delta, \theta_2) - f(x;\theta_1, \theta_2)] \mathrm{sgn}[\delta(x) - g(\theta_1, \Delta)] d\mu$$
$$- \int k_1\Delta \mathrm{sgn}[\delta(x) - g(\theta_1+\Delta)] \frac{\partial f(x;\theta_1+\Delta, \theta_2)}{\partial \theta_2} d\mu$$
$$= \int [f(x;\theta_1+\Delta, \theta_2) - f(x;\theta_1, \theta_2) - k_1\Delta\partial f(x;\theta_1+\Delta, \theta_2)/\partial\theta_2]$$
$$\mathrm{sgn}[\delta(x) - g(\theta_1+\Delta)] d\mu \tag{2.6}$$

where $F_\delta(y;\theta)$ represents the distribution function of the estimator $\delta(X)$.
Now multiplying (2.2) by $k_2\Delta$ and subtracting it from (2.6), we obtain the identity ;

$$2[F_\delta(g(\theta_1+\Delta);\theta_1,\theta_2)-F_\delta(g(\theta_1),\theta_1,\theta_2)]$$
$$=\int[f(x;\theta_1+\Delta,\theta_2)-f(x;\theta_1,\theta_2)-k_1\Delta\frac{\partial f(x;\theta_1+\Delta,\theta_2)}{\partial\theta_2}$$
$$-k_2\Delta f(x;\theta_1+\Delta,\theta_2)]\cdot\operatorname{sgn}[\delta(x)-g(\theta_1+\Delta)]d\mu. \tag{2.7}$$

Following lemma summarizes above result in a form which will be more convenient for the derivation of lower bound.

Lemma 1. If $\delta(x)$ is a median-unbiased estimator of $g(\theta_1)$, we have the identity ;

$$2[F_\delta(g(\theta_1+\Delta);\theta_1,\theta_2)-F_\delta(g(\theta_1);\theta_1,\theta_2)]$$
$$=\int|f(x;\theta_1+\Delta,\theta_2)-f(x;\theta_1,\theta_2)-k_1\Delta\partial f(x;\theta_1+\Delta,\theta_2)/\partial\theta_2$$
$$-k_2\Delta f(x;\theta_1+\Delta,\theta_2)|s_1s_2d\mu \tag{2.8}$$

where $s_1=\operatorname{sgn}[f(x;\theta_1+\Delta,\theta_2)-f(x;\theta_1,\theta_2)-k_1\Delta\partial f(x;\theta_1+\Delta,\theta_2)/\partial\theta_2-k_2\Delta f(x;\theta_1+\Delta,\theta_2)]$

and $s_2=\operatorname{sgn}[\delta(x)-g(\theta_1+\Delta)]$.

Proof. It follows immediately from (2.7).

Remark 1. Introduction of the extra centering parameter k_2 in (2.7) is the key difference between our approach and most of previous results which do not consider this possibility. This additional degree of freedom achieved by the introduction of the extra parameter k_2 will be exploited crucially in the derivation of the sharper lower bound.

In order to obtain an analogue of Cramer-Rao type inequality for the median-unbiased estimators, we now assume the following regularity conditions for the density $f(x;\theta)$;

R : 1) Θ is an open set in R^2.

2) $(\partial/\partial\theta_1)f(x;\theta)$ and $(\partial/\partial\theta_2)f(x;\theta)$ exist for every $\theta\in\Theta$,

3) $0<E_\theta|(\partial/\partial\theta_i)\log f(X;\theta)|<\infty$ for every $\theta\in\Theta$ $i=1,2$.

Under the regularity condition R, we can state the main result of this paper.

Theorem 1. Let $g(\theta_1)$ be a real-valued differentiable function of θ_1. Let $\delta(X)$ be a median-unbiased estimator of $g(\theta_1)$ having density function $f_\delta(\cdot;\theta)$ which is continuous at $g(\theta_1)$. Then under the regularity conditons R, we have the inequality ;

$$\frac{1}{2 \cdot f_\delta(g(\theta_1);\theta_1, \theta_2)} \geq |g'(\theta_1)| / I_1^*(\theta) \tag{2.9}$$

where $I_1^*(\theta) = \inf_{(k_1, k_2) \in R^2} \int |(\partial f(x;\theta)/\partial\theta_1) - k_1(\partial f(x;\theta)/\partial\theta_2) - k_2 f(x;\theta)| \, d\mu.$

Moreover if the support of the density $f(x;\theta)$ does not depend on θ, we can write $I_1^*(\theta)$ in the form

$$I_1^*(\theta) = \inf_{(k_1, k_2) \in R^2} E_\theta |(\partial \log f(X;\theta)/\partial\theta_1) - k_1(\partial \log f(X;\theta)/\partial\theta_2) - k_2|. \tag{2.10}$$

Proof. Dividing (2.8) by Δ and taking limits of both sides of (2.8) as $\Delta \to 0$ for fixed constants k_1, k_2, we have the inequality ;

$$2 f_\delta(g(\theta_1);\theta) |g'(\theta_1)| \leq \int |\partial f(x;\theta)/\partial\theta_1 - k_1(\partial f(x;\theta)/\partial\theta_2) - k_2 f(x;\theta)| \, d\mu. \tag{2.11}$$

Here we have used the chain rule and fact that $|s_1 \cdot s_2| \leq 1$. Now taking the infimum of (2.11) with respect to k_1, k_2, we get the result (2.9) immediately This completes the proof.

Remark 2. Note that the lower bound (2.9) is strictly greater than that of Sung (1993) unless $k_2 = 0$. See Example 2 in Section 4 in which we have $k_2 \neq 0$.

3. Optimality condition

In this section we define an optimal median-unbiased estimator $\delta(x)$ of $g(\theta_1)$ and find an useful optimality condition for the existance of the median-unbiased estimator under the regularity condition R.

Definition. A median-unbiased estimator $\delta(X)$ of $g(\theta_1)$ is said to be *optimal* if

$$\frac{1}{2 \cdot f_\delta g(\theta_1);\theta_1, \theta_2)} = |g'(\theta_1)| / I_1^*(\theta) \quad \text{holds for all } \theta \in \Theta .$$

As a first step for finding an optimal median-unbiased estimator, we first characterize the necessary and sufficient conditions for the attainability of the bound (2.9).

Theorem 2. Let the conditions of Theorem 1 be satisfied by the family of density functions $\{ f(x;\theta_1, \theta_2);\theta \in \Theta \}$. Then the following identity holds ;

$$2 \cdot f_\delta(g(\theta_1);\theta_1, \theta_2) \mid g'(\theta_1) \mid$$
$$= \int \mid (\partial f(x;\theta)/\partial\theta_1) - k_1(\partial f(x;\theta)/\partial\theta_2) - k_2 f(x;\theta) \mid s_1 s_2 s_3 \, d\mu \qquad (3.1)$$

where $s_1 = \text{sgn}[\,\partial f(x;\theta)/\partial\theta_1) - k_1 \partial f(x;\theta)/\partial\theta_2) - k_2 f(x;\theta)\,]$

$s_2 = \text{sgn}[\,\delta(x) - g(\theta_1)\,]$

$s_3 = \text{sgn}[\,g'(\theta_1)\,]$.

Moreover a median-unbiased estimator $\delta(x)$ is optimal if and only if for some constants k_1, k_2, the identity

$$\text{sgn}[\,(\partial f(x;\theta)/\partial\theta_1) - k_1(\partial f(x;\theta)/\partial\theta_2) - k_2 f(x;\theta)\,]$$
$$= \text{sgn}[\,\delta(x) - g(\theta_1)\,]\,\text{sgn}[\,g'(\theta_1)\,] \text{ holds a.e. } \mu^* \qquad (3.2)$$

where $d\mu^*/d\mu = \mid \partial f(x;\theta)/\partial\theta_1) - k_1 \partial f(x;\theta)/\partial\theta_2) - k_2 f(x;\theta) \mid$.

Proof. Applying differentiability condition to (2.8) we get the following identity immediately ;

$$2f_\delta(g(\theta_1);\theta_1, \theta_2) \cdot (g'(\theta_1))$$
$$= \int [\,(\partial f(x;\theta)/\partial\theta_1) - k_1(\partial f(x;\theta)/\partial\theta_2) - k_2 f(x;\theta)\,] s_1 \cdot s_2 d\mu. \qquad (3.3)$$

Then using the identity $x = \mid x \mid \text{sgn}(x)$ we can write (3.3) as follows

$$2f_\delta(g(\theta_1);\theta_1, \theta_2) \mid g'(\theta_1) \mid$$
$$= \int \mid (\partial f(x;\theta)/\partial\theta_1) - k_1(\partial f(x;\theta)/\partial\theta_2) - k_2 f(x;\theta) \mid s_1 s_2 s_3 \, d\mu$$
$$= \int s_1 s_2 s_3 \, d\mu^*.$$

This completes the proof because $s_1 s_2 s_3 = 1$ holds if only if $s_1 = s_2 s_3$ a.e. with respect to μ^*.

4. Examples

In this section, we give some examples which illustrate the usefulness of the optimality conditions given in Section 3.

Example 1. Let $X_1, X_2, \cdots, X_n$ be i.i.d. random variables with common distribution $N(\mu, \sigma^2)$. Let $(\theta_1, \theta_2)=(\mu, \sigma^2)$. Then we have the log-likelihood function $l(\theta_1, \theta_2)$;

$$l(\theta_1, \theta_2) = -(n/2)\log(2\pi\sigma^2) - \sum(x_i-\mu)^2/\sigma^2$$

and $$l_{\theta_1} = \partial l/\partial\theta_1 = \sum(x_i-\mu)/\sigma^2 = (n/\sigma^2)(\bar{x}-\mu)$$

$$l_{\theta_2} = \partial l/\partial\theta_2 = -1/2\sigma^2 + \sum(x_i-\mu)^2/\sigma^4.$$

Let $\delta(X)=\bar{X}$. Then $\delta(X)$ is a median-unbiased estimator of μ because $\bar{X}\sim N(\mu, \sigma^2/n)$. We claim it is an optimal median-unbiased estimator of μ. We note that for $k_1=0$, $k_2=0$,

$$\text{sgn}[\,l_{\theta_1} - k_1 l_{\theta_2} - k_2\,] = \text{sgn}[\,l_{\theta_1}\,] = \text{sgn}[\,(\bar{x}-\mu)\,] \text{ holds a.e.}$$

which verifies the optimality condition.

Next example consider the two-sample problem for censored reliability data for which we have non-zero values of k_1, k_2.

Example 2. (Two-sample Censored Reliability data) Let $X_1, X_2, \cdots, X_m$ and $Y_1, \cdots, Y_n$ be two independent random samples from the densities $f(x;\lambda_1)=\lambda_1\exp(-\lambda_1 x)$ and $f(y;\lambda_2)=\lambda_2\exp(-\lambda_2 y)$, $x, y>0$, $\lambda_1, \lambda_2>0$, respectively. Suppose we only observe first p and q order statistics of X's and Y's. Suppose $\theta_1=\lambda_1/\lambda_2$ and $\theta_2=\lambda_2$ respectively, and suppose we are interested in estimting θ_1 in presence of nuisance parameter θ_2. We claim that the estimator

$$\delta(X,Y)=C\cdot[\sum_{i=1}^{q} Y_{(i)}+(n-q)Y_{(q)}]/[\sum_{i=1}^{p} X_{(i)}+(m-p)X_{(p)}]$$

is an optimal median-unbiased estimator of θ_1 where C is an appropriate constant defined below. First we note that the log-likelihood function $l(\theta_1, \theta_2)$ of censored data is given by ;

$$l(\theta_1\theta_2) = p\log\theta_1\theta_2 - \theta_1\theta_2\{\sum_{i=1}^{p} x_{(i)} + (m-p)x_{(p)}\} + q\log\theta_2$$
$$-\theta_2\{\sum_{j=1}^{q} y_{(j)} + (n-q)y_{(q)}\}$$

and $$l_{\theta_1} = \partial l/\partial\theta_1 = p/\theta_1 - \theta_2\{\sum_{i=1}^{p} x_{(i)} + (m-p)x_{(p)}\}$$

$$l_{\theta_2} = \partial l / \partial \theta_2 = (p+q)/\theta_2 - \theta_1 \{ \sum_{i=1}^{m} x_{(i)} + (m-p) x_{(p)} \} - \{ \sum_{j=1}^{q} y_{(j)} + (n-q) y_{(q)} \}.$$

Now we note that

$$\mathrm{sgn}[\, l_{\theta_1} - k_1 l_{\theta_2} - k_2 \,] = \mathrm{sgn}[\, (\theta_2 - k_1 \theta_1) \{ \sum x_{(i)} + (m-p) x_{(p)} \}$$
$$- k_1 \{ \sum y_{(j)} + (n-q) y_{(q)} \} + q/\theta_1 - (p+q)/\theta_2 - k_2 \,].$$

If we let $k_1 = \theta_2/\theta_1 + C^{-1} \cdot \theta_2$ and $k_2 = p/\theta_2 - (p+q)/\theta_2$, then we have the identity ;

$$\mathrm{sgn}[\, l_{\theta_1} - k_1 l_{\theta_2} - k_2 \,] = \mathrm{sgn}[\, C^{-1} \cdot \theta_1 \{ \sum x_{(i)} + (m-p) x_{(p)} \}$$
$$- \{ \sum y_{(j)} + (n-q) y_{(q)} \} \,] = \mathrm{sgn}[\, \delta(x, y) - \theta_3 \,].$$

Now if we choose C as the median of $\delta(X, Y)/\theta_1$, then we get the result that $\delta(X, Y)$ is an optimal median-unbiased estimator immediately from the optimality condition.

References

[1] Alamo, B. J.(1964), *Estmacion en Mediana*, Trabajos de Estadistica, Vol. 15, 93－102.

[2] Lehmann, E.(1983), Theory of Point Estimation, Wiley New York.

[3] Sung, N. K., Stangenhaus, G. and David, H. T.(1990), *A Cramer-Rao Analogue for Median-Unbiased estimators*, Trabajos de esadistica, Vol. 5, 89－94.

[4] Sung, N. K.(1993), *Improving L_1 Information Bound in the presence of a Nuisance Parameter for Median-Unbiased estimator*. Journal of Korean Statistical, Society, Vol. 22－1, 1－11.

[5] So, B. S.(1993), *A Sharp Cramer-Rao type Lower-Bound for Median-Unbiased Estimators*, Ewha Womans University, unpublished manuscript.