Honam Mathematical J. 17(1995), No. 1,pp. 97–106

GENERALIZED CLOSE–TO–CONVEX FUNCTIONS

KHALIDA INAYAT NOOR

Mathematics Department, College of Science,
P.O. Box 2455, King Saud University, Riyadh 11451, Saudi Arabia.

Abstract

We introduce a new class of analytic functions in the unit disk which generalizes the concepts of close-to-convexity and of bounded boundary rotation, and study its various properties including its connection with other classes of analytic and univalent functions.

1. Introduction

Let V_k be the class of functions of bounded boundary rotation and K be the class of close-to-convex functions. Let R_k be the class of analytic functions with bounded radius rotation. A function $f \epsilon V_k$ if, and only if, $zf' \epsilon R_k$. It is clear that $R_2 = S^*$, the class of starlike functions and $V_2 = C$ is the class of convex functions.

DEFINITION 1.1. Let f ,with $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in $E = \{z : \mid z \mid < 1\}$ and $f'(z) \neq 0$. Then $f \epsilon T_k$, $k \geq 2$, if and only if there exists a function $glon V_k$ such that, for $z \epsilon E$,

$$(1.1) \qquad Re \frac{f'(z)}{g'(z)} > 0.$$

We note that $T_2 = K$
The class T_k has been introduced and discussed in some details in [1]. We now define the following.

Received October 31, 1994.
Keywords and Phrases: Close-to-convex, starlike, univalent, Bazilevic functions, coefficient result.
1991 AMS(MOS) Subject Classification: 30C45, 30A32

DEFINITION 1.2. Let f with $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in the unit disk E with $\frac{f(z).f'(z)}{z} \neq 0$, $z \epsilon E$. Then $f \epsilon T_k(a,\gamma)$, $Re a \geq 0$, $0 \leq \gamma \leq 1$ if, and only if, there exists a function $g \epsilon T_k$ such that, for $z \epsilon E$,

$$zf'(z) + af(z) = (a+1)z(g'(z))^{\gamma}. \tag{1.2}$$

We note that $T_k(0,1) = T_k$ and $T_2(0,1) = K$.

2. Preliminary Results

We shall give here the results needed to prove the main theorems in the preceding section.

LEMMA 2.1 [2]. *Let u and v denote complex variables, $u = u_1 + iu_2$, $v = v_1 + iv_2$ and let $\phi(u,v)$ be a complex-valued function that satisfies the following conditions:*
(i) $\phi(u,v)$ is continuous in a domain $D \subset \mathbb{C}^2$.
(ii) $(1,0) \epsilon D$ and $\phi(1,0) > 0$.
(iii) $Re(iu_2, v_1) \leq 0$ whenever $(iu_2, v_1) \epsilon D$ and $v_1 \leq -\frac{1}{2}(1+u_2^2)$.
If $p(z) = 1 + b_1 z + b_2 z^2 + \cdots$ is a function that is analytic in E such that

$$(p(z), zp'(z)) \epsilon D \text{ and } Re\{\phi(p(z), zp'(z))\} > 0$$

hold for all $z \epsilon E$, then $Re p(z) > 0$ in E.

LEMMA 2.2 [1]. *Let $g \epsilon T_k$. Then, with $z = re^{i\theta}$ and $\theta_1 < \theta_2$,*

$$\int_{\theta_1}^{\theta_2} Re\left\{\frac{(zg'(z))'}{g'(z)}\right\} d\theta > -\frac{k}{2}\pi.$$

LEMMA 2.3 [3]. *Let $f \epsilon R_k$. Then f is starlike for $|z| < r_0$, where r_0 is given by*

$$r_0 = \frac{1}{2}\left[k - \sqrt{k^2 - 4}\right]. \tag{2.1}$$

LEMMA 2.4 [4]. *Let $g_1 \epsilon V_k$. Then there exist two starlike functions s_1 and s_2 such that, for $z \epsilon E$,*

$$g_1'(z) = \left(\frac{s_1(z)}{z}\right)^{\frac{k}{4}+\frac{1}{2}} / \left(\frac{s_2(z)}{z}\right)^{\frac{k}{4}-\frac{1}{2}}.$$

LEMMA 2.5 [5]. *Let* $p(z) = 1 + b_1 z + \cdots$ *be analytic with* $|z| = r\epsilon(0,1)$, *and if, for* $\alpha, C, \theta_1, \theta_2$ *with* $\alpha \geq 1, Rec \geq 0, 0 \leq \theta_1 < \theta_2 \leq 2\pi$, *it is true that*

$$\int_{\theta_1}^{\theta_2} Re\left[p(z) + \frac{\alpha z p'(z)}{c\alpha + p(z)}\right] d\theta > -\pi,$$

then

$$\int_{\theta_1}^{\theta_2} Rep(z)d\theta > -\pi, \quad z = re^{i\theta}.$$

3. Main Results

THEOREM 3.1. *Let* $0 \leq \gamma_1 < \gamma_2 \leq 1$. *Then* $T_k(a,\gamma_1) \subset T_k(a,\gamma_2)$.

Proof. Let $f\epsilon T_k(a,\gamma_1)$. Then

$$\begin{aligned} zf'(z) + af(z) =& (a+1)z(g'(z))^{\gamma}, \quad g\epsilon T_k \\ =& (a+1)z(h'(z))^{\gamma}, \quad \text{where } h'(z) = (g'(z))^{\frac{\gamma_1}{\gamma_2}}. \end{aligned}$$

Now, since $G\epsilon T_k$, there exists a function $g_1\epsilon V_k$ such that, for $z\epsilon E$, $Re\frac{g'(z)}{g_1'(z)} > 0$. Let $G_1'(z) = (g_1'(z))^{\frac{\gamma_1}{\gamma_2}}$. It is easy to show that $G_1\epsilon V_k$.
Thus

$$\frac{h'(z)}{G_1'(z)} = \left(\frac{g'(z)}{g_1'(z)}\right)^{\frac{\gamma_1}{\gamma_2}},$$

and, since $\frac{\gamma_1}{\gamma_2} < 1$, we have $Re\frac{h'(z)}{G_1'(z)} > 0$, $z\epsilon E$ and this implies that $h\epsilon T_k$. Therefore $f\epsilon T_k(a,\gamma_2)$ and the proof is complete.

THEOREM 3.2. *Let* $f\epsilon T_k(a,\gamma)$, $Rea \geq 0$, $0 < \gamma \leq 1$. *Then, for* $0 \leq \theta_1 < \theta_2 \leq 2\pi$, $z = re^{i\theta}$.

$$\int_{\theta_1}^{\theta_2} Re\left\{p(z) + \frac{zp'(z)}{a + p(z)}\right\} d\theta > -\frac{k\gamma}{2}\pi,$$

where $p(z) = \frac{zf'(z)}{f(z)}$.

Proof. We have

$$zf'(z) + af(z) = (a+1)(g'(z))^{\gamma}, \quad g\epsilon T_k.$$

Differentiating logarithmically, we obtain

$$\frac{a+\frac{zf'(z))'}{f'(z)}}{1+a\frac{f(z)}{zf'(z)}} = \gamma\frac{(zg'(z))'}{g'(z)} + (1-\gamma),$$

and, with $p(z) = \frac{zf'(z)}{f(z)}$, we have

$$Re\left\{p(z) + \frac{zp'(z)}{a+p(z)}\right\} \geq \gamma\left[Re\frac{(zg'(z))'}{g'(z)}\right].$$

Now, on using Lemma 2.2, we obtain the required result:

Special Cases

(i) Let $f\epsilon T_k(a,\gamma)$ with $\gamma = \frac{2}{k}, k \geq 2$. Then, for $z = re^{i\theta}, 0 \leq \theta_1 < \theta_2 \leq 2\pi$,

$$p(z) = \frac{zf'(z)}{f(z)}, \quad \int_{\theta_1}^{\theta_2} Re\left\{p(z) + \frac{zp'(z)}{a+p(z)}\right\} d\theta > -\pi.$$

(ii) Let $zF' = f\epsilon T_k(a,\gamma), \gamma = \frac{2}{k}, \frac{zf'(z)}{f(z)} = p(z)$. Then, using Lemma 2.5 with $c = a, \alpha = 1$ and theorem 3.2, we have, for $\theta_1 < \theta_2$,

$$\int_{\theta_1}^{\theta_2} Re\frac{(zF'(z))'}{F'(z)} d\theta > -\pi, \quad z = re^{i\theta}.$$

This implies that F is close-to-convex and hence univalent, see [6].

THEOREM 3.3. *Let $f\epsilon T_k(0,\gamma)$. Then, with $z = re^{i\theta}$ and $0 \leq \theta_1 < \theta_2 \leq 2\pi$,*

$$\int_{\theta_1}^{\theta_2} Re\frac{(zf'(z))'}{f'(z)} d\theta > -\frac{\gamma k\pi}{2}.$$

Proof. $f\epsilon T_k(0,\gamma)$ implies that

$$zf'(z) = z(g'(z))^{\gamma}, g\epsilon T_k$$

This gives us

$$\frac{(zf'(z))'}{f'(z)} = \frac{\gamma(zg'(z))'}{g'(z)} + (1-\gamma)$$

Now the required result follows on using Lemma 2.2.

REMARK 3.1. From a necessary and sufficient condition for f to be univalent in E due to Kaplan [6], we note that $f \epsilon T_k(0,\gamma)$ is univalent E for $2 \leq k \leq \frac{2}{\gamma}$.

REMARK 3.2. Goodman [7] defines the class $K(\beta)$ of close-to-convex functions of higher order as follows.

Let f with $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in E and $f'(z) \neq 0$. Then, for $\beta \geq 0$, $f \epsilon K(\beta)$ if, for $z = re^{i\theta}$ and $\theta_1 < \theta_2$

$$\int_{\theta_1}^{\theta_2} Re\left\{\frac{(zf'(z))'}{f'(z)}\right\} d\theta > -\beta\pi.$$

We note that $T_k(0,\gamma) \subset K\left(\frac{\gamma k}{2}\right)$ and thus many of the results proved in [7] for the general class $K(\beta)$ hold also for the class $T_k(0,\gamma)$ with suitable choices of k and γ. From the definition 1.2, we immediately have the following.

THEOREM 3.4. *(Integral Representation) A function $f \epsilon T_k(a,\gamma)$ if and only if there exists a function $F \epsilon T_k(\infty,\gamma)$ such that*

$$f(z) = \frac{(a+1)}{z^a} \int_0^z t^{a-1} F(t) dt. \tag{3.1}$$

We now investigate the coefficient problem for the class $T_k(a,\gamma)$.

THEOREM 3.5. *Let $f \epsilon T_k(a,\gamma)$ and be given by $f(z) = z \sum_{n=2}^{\infty} a_n z^n$. Then, for $n \geq 2, \frac{1}{2} < \gamma \leq 1$,*

$$| a_n | \leq c(k,\gamma) \left| \frac{a+1}{1+\frac{a}{n}} \right| n^{\frac{\gamma k}{2}+2\gamma-2},$$

where $c(k,\gamma)$ is a constant and depends only on k and γ. The index $\left(\frac{k\gamma}{2}+2\gamma-2\right)$ is best possible.

Proof. Since $f \epsilon T_k(a,\gamma)$, we can write

$$zf'(z) + af(z) = (a+1)z(g'(z))^\gamma, g \epsilon T_k.$$

Let $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$. Then, with $z = re^{i\theta}$

$$(3.2)\qquad \begin{aligned} |(n+a)a_n| &= \frac{1}{2\pi r^n}\int_0^{2\pi} |zf'(z)+af(z)|\,d\theta \\ &= \frac{|a+1|}{2\pi r^{n+\gamma-1}}\int_0^{2\pi} |zg'(z)|^{\gamma}\,d\theta, \quad g\epsilon T_k. \end{aligned}$$

Using Lemma 2.4 and definition 1.1, we have

$$(3.3)\qquad zg'(z) = \frac{(s_1(z))^{\frac{k}{4}+\frac{1}{2}}}{(s_2(z))^{\frac{k}{4}-\frac{1}{2}}} h(z),$$

where $s_1, s_2 \epsilon S^*$ and $Reh(z) > 0, z\epsilon E$. Now we define, for $\gamma > \frac{1}{2}, z = re^{i\theta}$,

$$\begin{aligned} I_\gamma(r) &= \int_0^{2\pi} |zg'(z)|^{\gamma}\,d\theta \\ &= \int_0^{2\pi} \left|\frac{(s_1(z))^{(\frac{k}{4}+\frac{1}{2})\gamma}}{(s_2(z))^{(\frac{k}{4}-\frac{1}{2})\gamma}}\right| |h(z)|^{\gamma}\,d\theta, \quad \text{using (3.3).} \end{aligned}$$

Using well-known distortion theorems for the starlike function s_2 and then applying Schwarts inequality, we have

$$\begin{aligned} I_\gamma(r) &\le \left(\frac{4}{r}\right)^{(\frac{k}{4}-\frac{1}{2})\gamma} \left(\int_0^{2\pi} |s_1(z)|^{(\frac{k}{2}+1)\gamma}\,d\theta\right)^{\frac{1}{2}} \left(\int_0^{2\pi} |h(z)|^{2\gamma}\,d\theta\right)^{\frac{1}{2}} \\ &\le \left(\frac{4}{r}\right)^{(\frac{k}{4}-\frac{1}{2})\gamma} \left(\int_0^{2\pi} \frac{d\theta}{|1-re^{i\theta}|^{(k+2)\gamma}}\right)^{\frac{1}{2}} \left(\int_0^{2\pi} \frac{|1+re^{i\theta}|^{2\gamma}}{|1-re^{i\theta}|^{2\gamma}}\,d\theta\right)^{\frac{1}{2}}, \end{aligned}$$

by subordination.
Hence, for $\gamma > \frac{1}{2}, k \ge 2$, we have

$$(3.4)\qquad I_\gamma(r) \le c(k,\gamma)/(1-r)^{\frac{k\gamma}{2}+2\gamma-1},$$

where $c(k,\gamma)$ is a constant depending upon k and γ only.

Taking $r = \left(1-\frac{1}{n}\right)$, we obtain the required result from (3.2) and (3.4). The function $f_0 \epsilon T_k(a,\gamma)$ defined by

$$zf_0'(z) = af_0(z) = (a+1)z(g'(z))^{\gamma}$$

with

$$g_0(z) + \frac{1}{(k+2)}\left\{\left(\frac{1+z}{1-z}\right)^{\frac{k}{2}+1} - 1\right\} \epsilon T_k,$$

shows that the index $\left(\frac{k\gamma}{2}+2\gamma-2\right)$ is best possible.

THEOREM 3.6. *Let* $Re a \geq 0, 0 \leq \gamma \leq 1$. *Then* $T_2(a,\gamma) \subset T_2(\infty,\gamma)$.

Proof. Let $f \epsilon T_2(a,\gamma)$. Then there exist $G_2 \epsilon C$ and p_2 with $Re p_2(z) > 0, z \epsilon E$ such that

$$\begin{aligned} zf'(z) + af(z) &= (a+1)zp_2^{\gamma}(z)(G_2'(z))^{\gamma} \\ &= (a+1)zp_1(z)G_1'(z), \end{aligned} \tag{3.5}$$

where $p_1(z) = p_2^{\gamma}(z)$, $G_1'(z) = (G_2'(z))^{\gamma}$. It can be easily seen that $Re p_1(z) > 0$, and $G_1 \epsilon C$ in E.

Define $G(z)$ such that

$$zG'(z) + aG(z) = (a+1)G_1(z), \quad G_1 \epsilon C,$$

or

$$G(z) = \frac{a+1}{z^a} \int_0^z t^{a-1} G_1(t)dt \tag{3.6}$$

Since $G_1 \epsilon C$, it follows that the function G defined by (3.6) also belongs to C.

Now, from the definition, it follows directly the $f \epsilon T_2(\infty,\gamma)$ if and only if it may be represented as

$$f(z) = p(z)zG'(z), \quad G \epsilon C, \quad Re p(z) > 0, \text{ in } E.$$

Let

$$f(z) = p(z)zG'(z), \tag{3.7}$$

where G is defined by (3.6)

We shall show that $Re p(z) > 0, z \epsilon E$, and thus proving $f \epsilon T_2(\infty,\gamma)$.

After some computation, we have from (3.5), (3.6) and (3.7)

$$p(z) + \frac{zp'(z)}{a + p_0(z)} = p_1(z), \quad \text{where } Re p_1(z) > 0$$

and

$$Re p_0(z) = Re \frac{(zG'(z))'}{G'(z)} > 0 \quad \text{in } E.$$

We form the functional $\phi(u,v)$ by taking $p(z)=u$, and $zp'(z)=v$ such that $\phi(u,v)=u+\frac{v}{a+p_0}$.

We note that

(i) $\phi(u,v)$ is continuous in a domain $D\subset \mathbb{C}^2$ since $a+p_0(z)\neq 0$ in E. Here D is a region contained in $\{(p(z)), zp'(z)); z\epsilon E\}$.

(ii) At $z=0, u=p(0)=1$ and $v=0$, so the point $(1,0)\epsilon D$ and $\phi(1,0)=1>0$.

(iii)

$$\begin{aligned} Re\phi(iu_2,v_1) =&Re\left\{iu_2+\frac{v_1}{a+p_0}\right\}\\ =&v_1 Re\frac{1}{a+p_0}=\frac{v_1(a_1+t_1)}{(a_1+t_1)^2+(a_2+t_2)^2}, \end{aligned}$$

where $a_1=Rea\geq 0, t_1=Rep_0>0$.

Hence, for $v_1\leq -\frac{1}{2}(1+u_2^2)$,

$$Re\phi(iu_2,v_1)\leq \frac{-\frac{1}{2}(1+u_2^2)(a_1+t_1)}{(a_1+t_1)^2+(a_2+t_2)^2}\leq 0.$$

Thus $\phi(u,v)$ satisfies all the conditions of Lemma 2.1, and so $Rep(z)>0$ in E. Hence $f\epsilon T_2(\infty,\gamma)$ and the proof is complete.

THEOREM 3.7. *Let $f\epsilon T_k(a,1)$. The f is univalent for $|z|<r_0$, where r_0 is given by (2.1).*

Proof. Since $f\epsilon T_k(a,1)$, there exists a function $F\epsilon T_k(\infty,1)$ such that

$$f(z)=\frac{a+1}{z^a}\int_0^z t^{a-1}F(t)dt.$$

Now $f\epsilon T_k(\infty,1)$ implies that

$$\begin{aligned} F(z) =&zg'(z), \quad g\epsilon T_k\\ =&p(z)G_1(z), \quad G_1\epsilon R_k, \quad Rep(z)>0 \text{ in } E. \end{aligned}$$

Thus, with $a=m+in, m>0$, we have

$$f(z)=\frac{(m+1)+in}{z^{m+n}}\int_0^z t^m p(t)G_1(t)t^{in-1}dt. \tag{3.8}$$

We define

$$G(z)=z\left(\frac{G_1(z)}{z}\right)^{\frac{1}{m+1}}$$

Then it follows that $G \epsilon R_k$, and using Lemma 2.3, we see that $G \epsilon S^*$ for $|z| < r_0$, where r_0 is given by (2.1). Further, let

$$f_1(z) = \left[(m+1+in)\int_0^z G^{m+1}(t)p(t).t^{in-1}dt\right]^{\frac{1}{m+1+in}}$$

f_1 is a Bazelevic function for $|z| < r_0$ and hence univalent for $|z| < r_0$, see [8]. Therefore, for $|z| < r_0$, $\frac{f_1(z)}{z} \neq 0$. We note that

$$f_1(z) = z\left[\frac{f(z)}{z}\right]^{\frac{1}{a+1}}, \quad a = m + in.$$

This means that, for $\left(\frac{f(z)}{z}\right)^{\frac{1}{a+1}}$, it is possible to select a uniform branch which takes the value one for $z = 0$ and which is analytic for $|z| < r_0$ and thus we conclude that f is univalent in $|z| < r_0$, where r_0 is given by (2.1).

Special Case. From theorem 3.7, we see that $f \epsilon T_2(a,1)$ is univalent in E.

THEOREM 3.8. *Let $f \epsilon T_k(\infty, \gamma), \gamma \neq \frac{1}{2}$. Then the radius R of the circle which f maps onto a starlike domain is given by*

$$R = \frac{1}{2\gamma_1}\left[k_1 - \sqrt{k_1^2 - 4\gamma_1}\right],$$

where $k_1 = (k+2)\gamma, \gamma_1 = (2\gamma - 1)$.

Proof. $f \epsilon T_k(\infty, \gamma)$ implies that

$$f(z) = (zg'(z))^{\gamma}, \quad g \epsilon T_k$$

Differentiating logarithmically and using a result in [1] for $g \epsilon T_k$, we obtain

$$\begin{aligned} Re\left\{\frac{zf'(z)}{f(z)}\right\} &= Re\left\{\frac{\gamma(zg'(z))'}{g'(z)} + (1-\gamma)\right] \\ &\geq \frac{\gamma[r^2 - (k+2)r + 1]}{1-r^2} + (1-\gamma) \\ &= \frac{(2\gamma - 1)r^2 - (k+2)\gamma r + 1}{1-r^2}, \end{aligned}$$

and this gives us the required result.

The well-known coefficients results for $g \epsilon K$ together with the definition of the class $T_2(a,1)$ yield at once the following.

THEOREM 3.9. *Let $f\epsilon T_2(a,1)$ and be given by $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. Then*

$$| a_n | \le \frac{n^2 \mid a+1 \mid}{\mid n+a \mid}, \quad n \ge 2.$$

These bounds are sharp as can be seen from the function $f_\theta \epsilon T_2(a,1)$ defined as follows

$$z f_\theta'(z) + a f_\theta(z) = (a+1) z g_\theta'(z),$$

where $g_\theta(z) = \frac{z}{(1-ze^{i\theta})^2}$.

Using the fact that $f\epsilon T_2(a,1)$ is univalent in E and Theorem 3.9 for n=2, we immediately have the following covering result for the class $T_2(a,1)$.

THEOREM 3.10. *Let $f\epsilon T_2(a,1)$. Then the disk E is mapped onto a domain that contains the disk*

$$| w | < 2 + \frac{4 \mid a+1 \mid}{\mid 2+a \mid}.$$

References

1. K.I. Noor, *On a generalization of close-to-convexity*, Internat. J. Math. Math. Sci. **6** (1983), 327-334.
2. S. Miller, *Differential inequalities and Caratheodory functions*, Bull. Amer. Math. Soc. **81** (1975), 79-81.
3. B. Pinchuk, *Functions with bounded boundary rotation*, I.J. Math. **10** (1971), 7-16.
4. D.A. Brannan, *On functions of bounded boundary rotation*, Proc. Edin. Math. Soc. **2** (1968/69), 329-347.
5. R. Parvatham and S. Radha, *On certain classes of analytic functions*, Ann. Polon. Math. **49** (1988), 31-34.
6. W. Kaplan, *Close-to-convex schlicht functions*, Mich. Math. J. **1** (1952), 169-185.
7. A.W. Goodman, *On close-to-convex functions of higher order*, Ann. Univ. Sci. Budapest, Eötöus Sect. Math. **25** (1972), 17-30.
8. J.B. Bazilevic, *On a case of integrability in quadratures of the Loewner-Kufaru Equation*, Mat. Sb. **37** (1955), 471-476 (Russian).