Honam Mathematical J. 17(1995), No. 1,pp. 49–55

SUBMANIFOLDS WITH CONSTANT MEAN CURVATURE VECTOR FIELDS

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1. Introduction

Let E_s^m be the m-dimensional pseudo-Euclidean space with the standard flat metric given by

$$\bar{g} = -\sum_{i=1}^{s} dx_i^2 + \sum_{j=s+1}^{m} dx_j^2,$$

where $(x_1, \cdots, x_m)$ is a rectangular coordinate system of E_s^m. For a positive number r and a point $c \in E_s^m$, we denote by $S_s^{m-1}(c,r)$ and $H_{s-1}^{m-1}(c,-r)$, the pseudo-Riemannian sphere and the pseudo-hyperbolic space defined respectively by

$$S_s^{m-1}(c,r) = \{x \in E_s^m \mid < x-c, x-c >= r^2\},$$

$$H_{s-1}^{m-1}(c,-r) = \{x \in E_s^m \mid < x-c, x-c >= -r^2\},$$

where $<,>$ denotes the indefinite inner product on the pseudo-Euclidean space. The point c is called the center of $S_s^{m-1}(c,r)$ and of $H_{s-1}^{m-1}(c,-r)$, respectively. We simply denote $S_s^{m-1}(0,1)$ and $H_{s-1}^{m-1}(0,-1)$ by S_s^{m-1} and H_{s-1}^{m-1}, respectively. In physics, S_1^{m-1} and E_1^m are known as de Sitter space-time and the Minkowski space-time, respectively. We denote by H^{m-1} the (connected) hyperbolic space, imbedded standardly in E_1^m, by

$$H^{m-1} = \{x \in E_1^m \mid < x, x >= -1 \text{ and } t > 0\},$$

where $t = x_1$ is the first coordinate of the Minkowski space-time E_1^m. A vector X in E_s^m is said to be space-like(respectively, time-like or light-like)

Received May 1, 1995.

This work was partially supported by BSRI program, Ministry of Education under the project number BSRI-95-1425

if $< X, X >> 0$ or $X = 0$(respectively,$< X, X >< 0$ or $< X, X >= 0$ with $X \neq 0$).

Let $x : M_t^n \to E_s^m$ be an isometric immersion of a pseudo-Riemannian manifold M_t^n into E_s^m. Denote the position vector field of the immersion $x : M_t^n \to E_s^m$ aslo by x. Then we have the well-known Beltrami equation

$$\Delta x = -nH, \tag{1.1}$$

where H is the mean curvature vector field of M_t^n in E_s^m.

In this paper, we investigate the submanifolds of Minkowski space-time E_1^m with constant mean curvature vector fields. As a result, we obtain the following results :

THEOREM A. *Let $x : M_t^n \to E_{t+1}^{n+2}$ be a complete isometric immersion with nonzero constant mean curvature vector field. Then*

(a) M_t^n is isometric to E_t^n.

(b) M_t^n is, up to congruences on E_{t+1}^{n+2}, reparametrized by $\bar{x} : E_t^n \to E_{t+1}^{n+2}$, $\bar{x}(u_1, \cdots, u_n) = (f(u_1, \cdots, u_n), u_1, \cdots, u_n, f(u_1, \cdots, u_n))$ for some function f on E_t^n with $\Delta f = -n$. Conversely, such submanifold has nonzero constant mean curvature vector field.

THEOREM B. *Let M^n be a submanifold of the Minkowski space-time E_1^m. If M^n has nonzero constant mean curvature vector field, then*

(a) the Ricci tensor of M^n is negative semi-definite,

(b) M^n is Ricci flat if and only if M^n lies in the $(n+2)$ -dimensional Minkowski space-time E_1^{n+2}.

2. Preliminaries

$E_{s,j}^m$ denotes the m-dimensional affine space with the metric $\bar{g}$ whose canonical form is

$$\begin{pmatrix} -I_s & 0 & 0 \\ 0 & I_{m-s-j} & 0 \\ 0 & 0 & O_j \end{pmatrix},$$

where I_k is the $k \times k$ identity matrix and O_j is the $j \times j$ zero matrix. The metric is non-degenerate if and only if $j = 0$, in which case we write E_s^m for $E_{s,0}^m$. In general, if M is an n-dimensional manifold whose tangent spaces have a metric of signature $(i, n-i-j, j)$ we write $M_{i,j}^n$, if $j = 0$, M_i^n is called a pseudo-Riemannian manifold, and if $j = i = 0$, M^n is called a Riemannian manifold. In this paper, we always assume that manifolds are connected.

Let $x : M_t^n \rightarrow E_s^m$ be an isometric immersion of a pseudo-Riemannian manifold M_t^n into E_s^m. $\nabla, \bar{\nabla}, D, h$ and A_ξ denote the Levi-Civita connection on (M_t^n, g) , the flat connection on $(E_s^m, \bar{g})$, the normal conection on the normal bundle of M_t^n, the second fundamental form and the Weingarten map with respect to ξ in the normal bundle, respectively. Then we have the following well-known formulae:

(a) $\bar{\nabla}_X Y = \nabla_X Y + h(X,Y)$,

(b) $\bar{\nabla}_X \xi = -A_\xi(X) + D_X \xi$,

(c) $g(A_\xi(X), Y) = \bar{g}(h(X,Y), \xi)$

for any X, Y tangent to M_t^n and ξ normal to M_t^n.

Note that $H = \frac{1}{n} trace\ h$ is called the mean curvature vector of the submanifold M_t^n of E_s^m. If $\{e_1, \cdots, e_n\}$ is a local orthonormal frame of the tangent bundle of M_t^n, then $trace\ h = \Sigma_{i=1}^n \epsilon_i h(e_i, e_i)$, where $\epsilon_i = g(e_i, e_i) = \pm 1$ for $i = 1, \cdots, n$. If $H \equiv 0$, then M_t^n is called a minimal submanifold of E_s^m. We may find the basic notations and formulae in [1,6].

3. Examples

In this section, we investigate the pseudo-Riemannian submanifolds in E_s^m with nonzero constant mean curvature vector field and give some examples.

LEMMA 3.1. *Let $x : M_t^n \rightarrow E_s^m$ be an isometric immersion with nonzero constant mean curvature vector $H \equiv x_0$. Then*

(a) *x_0 is a null vector, hence we have $t \leq s-1$ and $n \leq m-2$.*

(b) *if $m = n+2$, then we have $t = s-1$ and $h(X,Y) = \alpha(X,Y)x_0$ for some (0,2)type tensor α.*

Proof. (a) For any vector field X, Y tangent to M_t^n, we have

$$0 = Y < X, x_0 > = < \bar{\nabla}_Y X, x_0 > = < h(X,Y), x_0 > .$$

Hence we obtain

$$< x_0, x_0 > = < x_0, H > = \frac{1}{n} \Sigma_{i=1}^n \epsilon_i < x_0, h(e_i, e_i) > = 0,$$

where $\{e_1, \cdots, e_n\}$ is an orthonormal frame field of M_t^n.

(b) Choose a normal vector field y of M_t^n in E_{t+1}^{m+2} so that $< y, y > = 0$ and $< x_0, y > = -1$. Since $\{x_0, y\}$ spans the normal space of M_t^n in E_s^m, we have $h(X,Y) = \alpha(X,Y)x_0 + \beta(X,Y)y$ for some (0,2)type tensors α, β. Note that

$$\alpha(X,Y) = -< h(X,Y), y >= -< A_y(X), Y >$$

and

$$\beta(X,Y) = -< h(X,Y), x_0 >= -< \bar{\nabla}_X Y, x_0 >=< Y, \bar{\nabla}_X x_0 >= 0.$$

Thus the Lemma follows.

From Lemma 3.1, we know that no submanifolds of the Euclidean space have nonzero constant mean curvature vector, that is, if a submanifold of the Euclidean space has a constant mean curvature vector, then it is minimal in the Euclidean space.

Example 3.2. Let $M_{t_1}^{n_1}$ and $M_{t_2}^{n_2}$ be two submanifolds of $E_{s_1}^{m_1}$ and $E_{s_2}^{m_2}$ whose mean curvature vectors are H_1 and H_2, respectively. Then $M_{t_1}^{n_1} \times M_{t_2}^{n_2}$ is a submanifold of $E_{s_1+s_2}^{m_1+m_2}$ whose mean curvature vector H is equal to $\frac{1}{n_1+n_2}(n_1 H_1 \bigoplus n_2 H_2)$. Therefore, if H_1 and H_2 are constant and if either H_1 or H_2 is nonzero, then H is a nonzero constant vector field.

Let $L_{s-1}^n(x_0, \epsilon)$ be the null hypersection of S_s^{n+1}(for $\epsilon = 1$) or H_{s-1}^{n+1} (for $\epsilon = -1$) in E_s^{n+2} defined by

$$L_{s-1}^n(x_0, \epsilon) = \{x \in E_s^{n+2} | < x, x >=< x, x_0 >= \epsilon\},$$

where x_0 is a null vector in E_s^{n+2}. Then $L_{s-1}^n(x_0, \epsilon)$ is a flat totally umbilic submanifold of E_s^{n+2} with constant mean curvature vector field $H \equiv -\epsilon x_0$.

More generally, we obtain the following :

PROPOSITION 3.3. *Let $x : M_t^n \to E_s^{m+2}$ be an isometric immersion which also lies in S_s^{m+1} $\left(or H_{s-1}^{m+1}\right)$. Then the mean curvature vector field of M_t^n in E_s^{m+2} is a nonzero constant vector if and only if M_t^n is a minimal submanifold of a null hypersection $L_{s-1}^m(x_0, \epsilon)$ in E_s^{m+2}.*

Proof. Let H' be the mean curvature vector of M_t^n in S_s^{m+1} (or H_{s-1}^{m+1}), then we have $H = H' - \epsilon x$. Suppose that H is a nonzero constant vector x_0. Since x_0 is a null vector, we have $< H', H' >= -\epsilon$ and $< x, -\epsilon x_0 >= \epsilon$. Thus M_t^n is a minimal submanifold of the null hypersection $L_{s-1}^{m-2}(-\epsilon x_0, \epsilon)$.

Now suppose that M_t^n is a minimal submanifold of a null hypersection $L_{s-1}^{m-2}(x_0, \epsilon)$. Since $\{x, x_0\}$ spans the normal space of $L_{s-1}^{m-2}(x_0, \epsilon)$ in E_s^m, we have $H = \alpha x + \beta x_0$ for some functions α, β on M_t^n. Note that $< H, x >= -1$ and $< H, x_0 >= 0$. Thus we obtain $H \equiv -\epsilon x_0$.

4. Proofs of main theorems

Let M_t^n be a pseudo-Riemannian manifold of the pseudo-Euclidean space E_s^{n+2} with nonzero constant mean curvtre vector $H \equiv x_0$. Then Lemma 3.1 implies that x_0 is null, $s = t+1$ and $h(X,Y) = \alpha(X,Y)x_0$. Note that x_0 is normal to M_t^n at any point in M_t^n and $D_X x_0 = 0$ for any X tangent to M_t^n. If α is a unit speed curve in M_t^n with $\alpha(0) = q \in M_t^n$ and if $\xi(s)$ is a normal parallel translation of x_0 along α, then, by the uniqueness of the normal parallel translation, we have $\xi(s) \equiv x_0$. Thus we see that

$$\frac{d}{ds} < \alpha(s) - q, x_0 >=< \alpha'(s), x_0 >=< \alpha'(s), \xi(s) >= 0.$$

And from the fact $\alpha(0) = q$, we obtain

$$< \alpha(s) - q, x_0 >= 0.$$

That is, $\alpha(s) - q \in (\mathbf{R}x_0)^{\perp} = T_qM \bigoplus \mathbf{R}x_0$, where $\mathbf{R}x_0 = \{tx_0 | t \in \mathbf{R}\}$. Since M_t^n is connected, we know that $M \subset T_qM \bigoplus \mathbf{R}x_0 = E_{t,1}^{n+1}$ (cf. [3,5]).

Now we prove the main theorems :

Proof of Theorem A. (a) We may assume that $x_0 = (1, 0, \cdots, 0, 1)$. Then we have $E_{t,1}^{n+1} = T_qM \bigoplus Rx_0 = \{x_1, \cdots, x_{n+2})| x_1 = x_{n+2}\}$. Let $p : E_{t,1}^{n+1} \rightarrow E_t^n$ be the projection defined by $p(x_1, \cdots, x_{n+2}) = (x_2, \cdots, x_{n+1}$). Then p is a linear map which preserves the inner product. Let U be a coordinate neighborhood of M_t^n with coordinates $(x_1, \cdots, x_n)$, then

$$\begin{aligned} < \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} > &=< x_*(\frac{\partial}{\partial x_i}), x_*(\frac{\partial}{\partial x_j}) > \\ &=< p_*(x_*(\frac{\partial}{\partial x_i})), p_*(x_*(\frac{\partial}{\partial x_j})) > \\ &=< (p \circ x)_*(\frac{\partial}{\partial x_i}), (p \circ x)_*(\frac{\partial}{\partial x_j}) > . \end{aligned}$$

Thus we see that $p \circ x : M_t^n \rightarrow E_t^n$ is an isometric immersion. Since M_t^n is complete, we see that $p \circ x$ is an isometry (see [6] pp.201-202).

(b) Note that $x \circ (p \circ x)^{-1} : E_t^n \rightarrow E_{t,1}^{n+1}$ is an isometric immersion and that $p \circ (x \circ (p \circ x)^{-1})$ is the identity map. Thus we have

$$(x \circ (p \circ x)^{-1})(u_1, \cdots, u_n) = (f(u_1, \cdots, u_n), u_1, \cdots, u_n, f(u_1, \cdots, u_n)).$$

for some function f on E^n_t. Let $\bar{x}$ be the reparametrization $x \circ (p \circ x)^{-1}$ of M^n_t. From the Beltrami equation (1.1) we have

$$\Delta \bar{x} = (\Delta f, 0, \cdots, 0, \Delta f) = -nH = (-n, 0, \cdots, 0, -n).$$

Hence we see that f is a function on E^n_t satisfying $\Delta f = -n$.

The converse is obivious.

Proof of Theorem B. (a) From Lemma 3.1(b), we see that M^n is space-like. Let $\{e_1, \cdots, e_n\}$ be an orthonormal frame of M^n, then for any X, Y tangent to M^n we have

$$\begin{aligned} Ric(X,Y) =& tr(V \to R(X,V)Y) = \Sigma_{i=1}^n < R(X,e_i)Y, e_i > \\ =& \Sigma_{i=1}^n \{< \bar{R}(X,e_i)Y, e_i > + < h(X,Y), h(e_i,e_i) > \\ & - < h(X,e_i), h(Y,e_i) >\} \\ =& \Sigma_{i=1}^n \{< h(X,Y), h(e_i,e_i) > - < h(X,e_i), h(Y,e_i) >\}, \end{aligned}$$

where the third equality follows from Gauss equation. Thus we obtain

$$\begin{aligned} Ric(X,X) =& \Sigma_{i=1}^n \{< h(X,X), h(e_i,e_i) > - < h(X,e_i), h(X,e_i) >\} \\ =& - \Sigma_{i=1}^n < h(X,e_i), h(X,e_i) >, \end{aligned}$$

where the second equality follows from the fact $< x_0, h(X,Y) >= 0$.

On the other hand, let $\{x_0, y, e_{n+3}, \cdots, e_m\}$ be a pseudo-orthonormal frame of th normal bundle, that is, $\{x_0, y, e_{n+3}, \cdots, e_m\}$ is a frame of the normal b e satisfying the following :

$$< x_0, y >= -1, \quad < e_t, e_t >= 1 \text{ and otherwise zero}, \quad t = n+3, \cdots, m.$$

And let A_t denote A_{e_t} for $t = n+3, \cdots, m$. Then $< h(X,e_i), e_t >=< A_t(X), e_i >$ and $< h(X,e_i), x_0 >= 0$. Hence we obtain

$$h(X,e_i) = \alpha(X,e_i)x_0 + \Sigma_{t=n+3}^m < A_t(X), e_i > e_t.$$

Finally we have

$$(*) \qquad \begin{aligned} Ric(X,X) =& - \Sigma_{i=1}^n \Sigma_{t=n+3}^m < A_t(X), e_i >^2 \\ =& - \Sigma_{t=n+3}^m < A_t(X), A_t(X) > \leq 0. \end{aligned}$$

(b) Suppose that $Ric \equiv 0$, then by $(*)$ we see that $A_t(X) = 0$ for $t = n+3, \cdots, m$. Hence we have

$$h(X,Y) = \alpha(X,Y)x_0, \quad \bar{\nabla}_X e_i = \Sigma_{j=1}^n w_i^j(X)e_j + \alpha(X,e_i)x_0,$$

which implies that $\bar{\nabla}_X(e_1 \wedge \cdots \wedge e_n \wedge x_0) = 0$. Therefore for a fixed point $q \in M^n$, M^n lies in $T_qM^n \bigoplus Rx_0 = E^{n+1}_{0,1}$. Thus M^n lies in the $(n+2)$-dimensional Minkowski space-time E^{n+2}_1.

The converse follows from Theorem A.

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