Honam Mathematical J. 17(1995), No. 1,pp. 73–82

ON INVERSE LIMIT OF CONTINUA*

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1. Introduction

In 1920, Canster and Kuratowski [4] asked if nondegenerate homogenous continuum in $\mathbb{R}^2$ must be a simple closed curve. Subsequently Mazurkiewicz [14] asked if every continuum in $\mathbb{R}^2$ which is homeomorphic to each of its nondegerate subcontinua must be an arch, and they described the example of a nondegenerate hereditarily in decomposible continua. Bing [3] and Moise [15] answered the question in [15] negatively. And he also showed that most continua in $\mathbb{R}^n$ or Hilbert space are pseudo arcs. This chain of event and results are undoubtely responsible for the continuing interest in and development of the theory of arc like continua. In our study, we have constructed the special arc and indicated how to prove it is hereditarily indecomposible in $\sin(\frac{1}{x})$ continuum. A symmetric treatment of the arc, even if it was limited to the result mentioned above, would require space we do not have, instead, we shall devote this paper to some the general inverse limit theory of arc like continua.

2. Preliminaries

An inverse sequense is a *double sequence* $\{X_i, f_i\}_{i=1}^{\infty}$ of spaces X_i, called coordinate spaces, and continuous functions $f_i\colon X_{i+1} \to X_i$ called bonding maps. If $\{X_i, f_i\}_{i=1}^{\infty}$ is an inverse sequence, sometimes written

$$X_1 \xleftarrow{f_1} X_2 \xleftarrow{f_2} \cdots \xleftarrow{f_{i-1}} X_i \xleftarrow{f_i} X_{i+1} \xleftarrow{f_{i+1}} \cdots ,$$

then the inverse limit of $\{X_i, f_i\}_{i=1}^{\infty}$, denoted by $\varprojlim\{X_i, f_i\}_{i=1}^{\infty}$, is the subspace of the cartesian product space $\prod_{i=1}^{\infty} X_i$ denoted by

$$\varprojlim\{X_i, f_i\}_{i=1}^{\infty} = \{(x_i)_{i=1}^{\infty} \in \prod_{i=1}^{\infty} X_i\colon f_i(x_{i+1}) = x_i \text{ for all } i\}.$$

Received April 27, 1995.

* This was partially supported by Basic Science Research Institute Program, Ministry of Education. 1994.

Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of compact metric spaces such that $X_i \supset X_{i+1}$ for each $i = 1, 2, \cdots$, and let $X = \cap_{i=1}^{\infty} X_i$. If U is an open subset of X_i such that $U \supset X$, then there exists N such that $U \supset X_i$ for all $i \geq N$. In particular, if each $X_i \neq \phi$, then $X \neq \phi$ (and clearly, compact metric).

PROPOSITION 2.1. *If $\{X_i\}_{i=1}^{\infty}$ is a sequence of continua such that $X_i \supset X_{i+1}$ for each $i = 1, 2, \cdots$, and*

$$X = \bigcap_{i=1}^{\infty} X_i.$$

Then, X is a continuum.

Proof. Since X is a nonempty, compact metric space, it suffices to show that X is connected. Suppose that X is not connected. Then $X = A \cup B$ where A and B are disjoint, nonempty, closed (hence, compact) sets. Since X_1 is a normal space, there are disjoint open subsets V and W such that $A \subset V$ and $B \subset W$. Let $U = V \subset W$. Then $U \supset X_n$ for some n. Hence, $X_n = (X_n \cap V) \cup (X_n \cap W)$. Since $X_n \supset X = A \cup B$ and since $A \neq \phi$ and $B \neq \phi$, we see that $X_n \cap V \neq \phi$ and $X_n \cap W \neq \phi$. It now follows easily that X_n is not connected, a contradiction. Therefore, X is connected.

A continumm X is said to be ***decomposible*** provided that X can be written as the union of two proper subcontinuum. A continuum which is not decomposible is said to be indecomposible.

LEMMA 2.2. *Let $X_\infty = \varprojlim\{X_i, f_i\}_{i=1}^{\infty}$. Let A and B be compact subsets of X_∞, and let $c = A \cap B$. If, for each i, $\pi_i \colon X_\infty \to X_i$ is the i-th projection map and $c_i = \pi_i(A) \cap \pi_i(B)$, then c is $\varprojlim\{c_i, f_i \mid c_{i+1}\}_{i=1}^{\infty}$.*

Proof. Let c is $\varprojlim\{c_i, f_i \mid c_{i+1}\}_{i=1}^{\infty}$. Take each $x = (x_1, x_2, \cdots)$ in c. For each i, $\pi_i(x) = x_i \in c_i$ and $\pi_{i+1}(x) = x_{i+1} \in c_{i+1}$ such that, for all i, $f(x_{i+1}) = x_i$. Since $x \in X_\infty$ and $x \in c$, $f(x_{i+1}) = x_i$. So $x \in c_\infty$and $c \subset c_\infty$. Take each $y = (y_1, y_2, \cdots)$ in c_∞. Then, for each i,

$$y_i \in c_i = \pi_i(A) \cap \pi_i(B) \supset \pi(A \cap B)$$

$$\begin{aligned} y \in \pi_i^{-1}(y_i) \subset \pi_i^{-1}(c_i) &= \pi_i^{-1}(\pi_i(A) \cap \pi_i(B)) \\ &\subset \pi_i^{-1}(\pi_i(A)) \cap \pi_i^{-1}(\pi_i(B)) = A \cap B = c. \end{aligned}$$

Therefore $c = c_\infty = \varprojlim\{c_i, f_i \mid c_{i+1}\}_{i=1}^{\infty}$.

THEOREM 2.3. *An inverse limit of arcs cannot contain a simple closed curve.*

Proof. Let X_∞ be an inverse limit of arcs. Assume X_∞ contains a simple closed curve. Then, by 2.2, $c_i = \pi_i(A) \cap \pi_i(B)$ and $c = \varprojlim\{c_i, f_i \mid c_{i+1}\}_{i=1}^{\infty}$. But, since c is a simple closed curve, $S^1 = \{x \in \mathbb{R}^2 : \|x\| = 1\}$ is homeomorphic to c. There exist p', q' in S^1 where p' is an interior point of S^1 and q' is an end point of S^1, such that $p' = q'$. Put $f(p') = p = (p_1, p_2, \cdots)$ and $f(q') = q = (q_1, q_2, \cdots)$ in c. Then, for each i, $p_i = q_i$ and $\pi_i(p)$ and $\pi_i(q)$ are in $\pi_i(c) = c_i$. But, since $c_i \subset X_i$ and X_i is an arc, for all i, $p_i \neq q_i$ in X_i. Hence $p = (p_1, p_2, \cdots) \neq q = (q_1, q_2, \cdots)$ in $\varprojlim\{c_i, f_i \mid c_{i+1}\}_{i=1}^{\infty}$. But, $c = \varprojlim\{c_i, f_i \mid c_{i+1}\}_{i=1}^{\infty}$ and $p = (p_1, p_2, \cdots) = q = (q_1, q_2, \cdots)$ in c.

PROPOSITION 2.4. *Consider the situation in the diagram below where $X_\infty = \varprojlim\{X_i, f_i\}_{i=1}^{\infty}$, $Y_\infty = \varprojlim\{Y_i, g_i\}_{i=1}^{\infty}$, each rectangle is commutative ; $\phi_i \circ f_i = g_i \circ \phi_{i+1}$ for each i. Then the following hold :*

(1) *ϕ_∞ maps X_∞ into Y_∞,*
(2) *if each ϕ_i is continuous, then ϕ_∞ is continuous,*
(3) *if each ϕ_i is one to one, then ϕ_∞ is one to one,*
(4) *if each ϕ_i maps continuously onto Y_i and if X_i is a compact metric space, then ϕ_∞ maps X_∞ onto Y_∞.*

$$\begin{array}{ccccccccccccc}
X_1 & \xleftarrow{f_1} & X_2 & \xleftarrow{f_2} & \cdots & \longleftarrow & X_i & \xleftarrow{f_i} & X_{i+1} & \xleftarrow{f_{i+1}} & \cdots & \longleftarrow & X_\infty \\
\downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_2} & & & & \downarrow{\scriptstyle \phi_i} & & \downarrow{\scriptstyle \phi_{i+1}} & & & & \downarrow{\scriptstyle \phi_\infty} \\
Y_1 & \xleftarrow[g_1]{} & Y_2 & \xleftarrow[g_2]{} & \cdots & \longleftarrow & Y_i & \xleftarrow[g_i]{} & Y_{i+1} & \xleftarrow[g_{i+1}]{} & \cdots & \longleftarrow & Y_\infty
\end{array}$$

Proof. Define $\phi_\infty : X_\infty \to Y_\infty$ by $\phi_\infty((x_i)_{i=1}^{\infty}) = (\phi_i(x_i)_{i=1}^{\infty})$ for each $(x_i)_{i=1}^{\infty} \in X_\infty$.

(1) ; Since, for all i, $\phi_i : X_i \to Y$ is a map, for each $x = (x_1, x_2, \cdots)$, $\phi_i(x_i) \subset Y_i$ and $g_n\phi_{i+1}(x_{i+1}) = \phi_n f_n(x_{i+1}) = \phi_n(x_n)$. Hence $(\phi_1(x_1), \phi_1(x_1), \cdots) \in Y_\infty$.

(2) ; $\pi\phi_i$ is continuous, since ϕ_i is continuous.

(3) ; Let $y_1 = (y_1^1, y_2^1, \cdots) \neq y_2 = (y_1^2, y_2^2, \cdots)$ in Y_∞. Then, for each i, since ϕ_i is continuous, $\phi_i^{-1}(y_i^1) \neq \phi_i^{-1}(y_i^2)$. Hence $y_1 = (x_1^1, x_2^1, \cdots) \neq (x_1^2, x_2^2, \cdots)$ in Y_∞.

(4) ; Fix $y = (y_1, y_2, \cdots) \in Y_\infty$. Put $Z_i = \phi_i^{-1}(y_i)$ in X_i. Since f is continuous and X_i is compact, $f(X_i)$ is compact and $\phi_i^{-1}(y_i) = Z_i$ is compact. Hence $\phi_\infty^{-1}(y_i) = \{(x_1, x_2, \cdots) | x_i \in Z_i\}$ and since x_{i+1} in Z_{i+1} and in $\phi_i^{-1}(y_{i+1})$, $\phi_i f(x_{i+1}) = \phi_i(x_i) = y_{i-1} = g_i \phi_{i+1}(x_{i+1})$. $\phi_i f(x_{i+1}) = g_i \phi_{i+1}(x_{i+1}) = y_{i-1}$. $\phi_i^{-1}(y_{i+1}) = Z_{i-1} \ni f(x_{i+1})$.

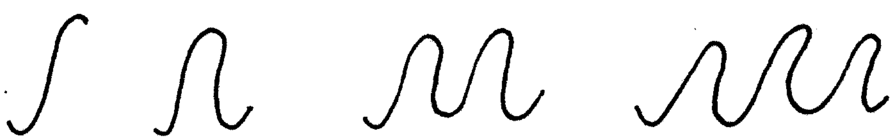
Let X and Y be metric spaces and let $f\colon X \to Y$. Then, f is called an ϵ-map provided that f is continuous and the diameter of $f^{-1}(f(x)) < \epsilon$ for all $x \in X$.

3. Inveerse limits and Main theorems

The $\sin(\frac{1}{x})$-continuum is the closure $\overline{W}$ of W where $W = \{(x_i, \sin(\frac{1}{x})) \in \mathbb{R}^2 : 0 < x \leq 1\}$.

THEOREM 3.1. *(a) If X is a $\sin(\frac{1}{x})$-continuum, then X is arc-like. (b) If $X = \sin(\frac{1}{x})$-continuum and the commutative diagram below, where each $X_i = X$, each $Y_i \subset Y_{i+1}$ is a particular arc in W begin at $(1, \sin(1))$, each f_i is the identity map and each g_i is bonding maps and ϕ_i is a natural horizontal projection, then X is homeomorphic to Y_∞.*

$$\begin{array}{ccccccccccccc}
X_1 & \xleftarrow{f_1} & X_2 & \xleftarrow{f_2} & \cdots & \xleftarrow{f_{i-1}} & X_i & \xleftarrow{f_i} & X_{i+1} & \xleftarrow{f_{i+1}} & \cdots & \longleftarrow & X_\infty \\
\downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_2} & & & & \downarrow{\scriptstyle \phi_i} & & \downarrow{\scriptstyle \phi_{i+1}} & & & & \downarrow{\scriptstyle \phi_\infty} \\
Y_1 & \xleftarrow[g_1]{} & Y_2 & \xleftarrow[g_2]{} & \cdots & \xleftarrow[g_{i-1}]{} & Y_i & \xleftarrow[g_i]{} & Y_{i+1} & \xleftarrow[g_{i+1}]{} & \cdots & \longleftarrow & Y_\infty
\end{array}$$



Proof. (a) ; Define $f_\epsilon\colon X \to [0,1]$ by

$$\begin{cases} f_\epsilon(x) = (x, \sin\frac{1}{x}) = \frac{x}{2} + \frac{1}{2}, & 0 < x \leq 1 \\ f_\epsilon(t) = (0,1) = \frac{t}{4} + \frac{1}{4}, & -1 \leq t \leq 1 \ . \end{cases}$$

Then f_ϵ is one to one and continuous. (b) ; Using proposition 2.4, we can define $\phi_\infty\colon X_\infty \to Y_\infty$, $X_\infty \subset \prod X_i$ and $Y_\infty \subset \prod Y_i$. Let $x = (x_1, x_2, \cdots)$, $y = (y_1, y_2, \cdots)$ in X_∞. Assume $\phi_\infty(x) \neq \phi_\infty(y)$. Then, for some i, $\phi_i(x_i) \neq \phi_i(y_i)$ in Y_i such that $\phi_{i-1} f \phi^{-1}(\phi_i(x_i)) = g(\phi_i(x_i))$ and $\phi_{i-1} f \phi^{-1}(\phi_i(y_i)) = g(\phi_i(y_i))$. But $g(\phi_i(x_i)) \neq g(\phi_i(y_i))$ and so $\phi_{i-1} f \phi^{-1}(\phi_i(y_i)) \neq g(\phi_i(x_i))$ in X_i. Hence $x_i \neq y_i$ in X_i and $\phi^{-1}(x_i)$ is the i-th coordinate of $x = (x_1, x_2, \cdots)$ in X_∞ and $\phi^{-1}(y_i)$ is the i-th coordinate of $y = (y_1, y_2, \cdots)$ in X_∞. Therefore $x \neq y$ and ϕ_∞ is one to one. $Y_\infty = X_\infty \mid \prod Y_i$ such that $Y_\infty \subset X_\infty$. But since $\phi_\infty \colon Y_\infty \to X_\infty$ is one to one, continuous, $Y_\infty \cong X_\infty$. So X can be represented by Y_∞.

LEMMA 3.2. *Suppose that A_0 is the convex arc in the plane $\mathbb{R}^2$ from $(0,1)$ to $(0,0)$ and, for each $n = 1, 2, \cdots$, A_n is the convex arc in the plane $\mathbb{R}^2$ from $(0,1)$ to $(0, 2^{-n+1})$ and, for each $n = 1, 2, \cdots$, $X_i = A_0 \cup (\cup_{n=1}^{i+1} A_n)$, $Y_i = \cup_{n=1}^{i+1} A_n$ and $f_i \colon X_{i+1} \to X_i$ and $g_i \colon Y_{i+1} \to Y_i$ are the natural maps, f_1 mapping A_{i+1} linearly onto A_0, g_i mapping A_{i+1} linearly onto A_i and, in both case, leaving all other points fixed. Then the two inverse limits $X_\infty = \varprojlim \{X_i, f_i\}_{i=1}^\infty$, $X_\infty = \varprojlim \{X_i, f_i\}_{i=1}^\infty$ are homeomorphic.*

Proof. Define $\phi_i \colon X_i \to Y_i$ by for each x_i in X_i,

$$\phi_i(x) = \begin{cases} x_i & \text{if } x_i \in Y_i \\ (0,1) & \text{if } x_i \notin Y_i \end{cases}$$

Then ϕ_i and ϕ_i^{-1} are continuous function. Define $\phi_\infty \colon X_\infty \to Y_\infty$ by for each $x = (x_1, x_2, \cdots)$ in X_∞, $\phi_\infty(x) = (\phi_1(x_1), \phi_2(x_2), \cdots)$. To be prove that ϕ_∞ is homeomorphism, we will show that ϕ_∞ is one to one. (ϕ_∞ is onto ; for each y in Y_∞, $y \in X_\infty (\supset Y)$, ϕ_∞, ϕ_∞^{-1} are continuous ; ϕ_∞, ϕ_∞^{-1} are continuous)

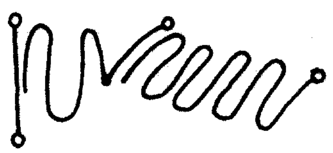
Assume $\phi_\infty((x_i)_{i=1}^\infty) \neq (\phi_i(y_i)_{i=1}^\infty)$. Then, for some i, $x_i \neq y_i$ and so $x \neq y$ in X_∞.

A continuum X is said to be hereditarily decomposible provided that each nondegenerate subcontinuum of X is decomposible. A subset D of a continuum X is said to be continuumwise dense in X if $D \cap A \neq \phi$, for all A, is nondegenerate subcontinua A of X.

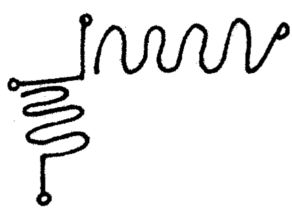
EXAMPLE 3.3. *Construct a decomposible continuum which contains no arc.*

Proof. We start the construction by letting X_1 be the $\sin(\frac{1}{x})$-continuum. Let $p = (1, \sin(1))$, $q = (0,1)$ and $r = (0,-1)$. Let $D_1 = \{x_n^1 \colon n = 1, 2, \cdots\}$ is countable subset of $X_1 - \{p, q, r\}$ such that D_1 is continuumwise dense in X_1. Note that if $J = X_1 - W (W = \sin(\frac{1}{x})$-continuum), then

$\overline{D_1 \cap J} = J$. Now let X_2 be the continuum obtained from X_1 by "replacing" the point x_1^1 with a copy Y of X_1, by this we mean that X_2 is one of the two continua drowing below. Let K denote the arc in Y corresponding to the arc J in X_1. Let f_1 from X_2 onto X_1 be the natural map which takes K to x_1^1 and is a homeomorphism from $X_2 - K$ onto $X_1 - \{x_1^1\}$. Let s and t denote the two end points of K. Let $D_2 = \{x_n^2 : n = 1, 2, \cdots\}$ be cotinuumwise dense in X_2 such that D_2 misses $f_1^{-1}(\{p, q, r\})$, $\{s, t\}$, and $f_1^{-1}(x_2^1)$ for all $n \geq 2$. From x_3 from x_2 by *replacing* each of the two points x_1^2 and $f_1^{-1}(x_2^1)$ with a copy of X_1. The map f_2 from X_3 onto X_2 is defined in a manner similar to the way f_1 was defined. We obtain X_4 from X_3 by a procedure similar to the one used to obtain X_3 from X_2, this time making sure that copies of X_1 are inserted in X_3 at the first enumerated point x_1^3 of D_3 and at each of the two points $f_2^{-1}(x_2^2)$ and $(f_1 \circ f_2)^{-1}(x_3^1)$. Continuing in this fashion, we obtain the inverse sequence $\{X_i, f_i\}_{i=1}^{\infty}$. Then, to be proved that $\{X_i, f_i\}_{i=1}^{\infty} = X_\infty$ is a decomposible continuum which contains no arc, we will show that X_∞ is arcless and each subcontinuum A of X_∞ has a cut point. X_∞ is arcless ; $X_\infty \cong \lim_{i \to \infty} X_i$ by theorem 3.1, but since $\lim_{i \to \infty} X_i$ is arcless. Each $A \subset X_\infty$ has a cut point ; define $\pi_i^{-1} : X_\infty \to X_i$: natural projection (take i large enough). Then, for each $A \subset X_\infty$, $\pi_i^{-1}(A)$ contains some cut point. Therefore A is decomposible.



X_2 when $x_1^1 \in W$



X_2 when $x_1^1 \in J$

THEOREM 3.4. *Let Y be a compact matric space, and let $\{Y_i\}_{i=1}^{\infty}$ be a sequence of compact subset of Y, such that, for each $i = 1, 2, \cdots$, there are continuous functions g_i from Y_{i+1} onto Y_i and ϕ_i from Y onto Y_i such that $g_i \circ \phi_{i+1} = \phi_i$; if the sequence $\{\phi_i\}_{i=1}^{\infty}$ converges uniformly to the identity map on Y, then Y is homeomorphic to $\underleftarrow{\lim}\{Y_i, g_i\}_{i=1}^{\infty}$.*

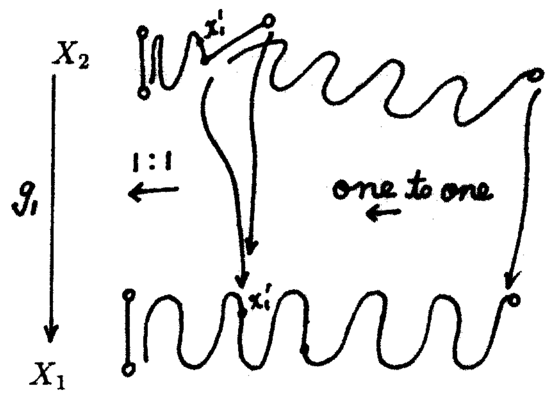
Proof. Define $\pi : Y_\infty \to Y$: by $\pi(x) = \lim_{i \to \infty} x_i$ where $x = (x_1, x_2, \cdots)$ in Y_∞ and $\pi_n : Y_\infty \to Y_n$ by natural projection. Then π is one to one ; if for $x = (x_1, x_2, \cdots)$, $y = (y_1, y_2, \cdots) \in Y_\infty$, $\pi(x) \neq \pi(y)$, then there exists i such that $x_i \neq y_i$ and $x_{i+1} \neq y_{i+1}$, $x_{i+1} \neq y_{i+1}, \cdots$. So $x \neq y$ in X_∞, onto ; for each $y \in Y$, there exists $(x_1, x_2, \cdots, y_1, y_2, \cdots)$ in Y_∞ such that $\pi(x_1, x_2, \cdots, y_1, y_2, \cdots) = y$, continuous ; u is open in Y, $\pi^{-1}(u) = \pi_n^{-1}\phi_n(u)$, n : large enough, by the same way π^{-1} is continuous. Therefore $Y \cong \{Y_i, g_i\}_{i=1}^{\infty}$.

LEMMA 3.5. *X_∞ which is constructed in 3.3 is embedable in $\mathbb{R}^2$.*

Proof. Put $X = \lim_{i\to\infty} X_i$ where X_i is a subset of X and $X_i \subset X_{i+1}$.

$$\begin{array}{ccccccccccccc}
X_1 & \xleftarrow{g_1} & X_2 & \xleftarrow{g_2} & \cdots & \xleftarrow{g_{i-1}} & X_i & \xleftarrow{g_i} & X_{i+1} & \xleftarrow{g_{i+1}} & \cdots & \longleftarrow & X_\infty \\
\phi_1\uparrow & & \phi_2\uparrow & & & & \phi_i\uparrow & & \phi_{i+1}\uparrow & & & & \uparrow 1_X \\
Y_1 & \xleftarrow[1_X]{} & Y_2 & \xleftarrow[1_X]{} & \cdots & \xleftarrow[1_X]{} & Y_i & \xleftarrow[1_X]{} & Y_{i+1} & \xleftarrow[1_X]{} & \cdots & \longleftarrow & Y_\infty
\end{array}$$

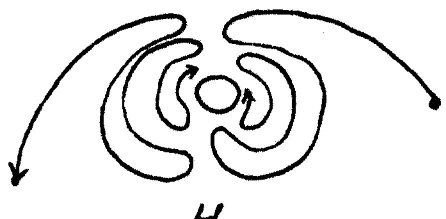
Define g_1:



By the same way, define g_i and ϕ_i. Then ϕ_i converges uniformly to 1_X. Then, by 3.4, $X_\infty \cong X$. Hence X_∞ is embedable in $\mathbb{R}^2$.

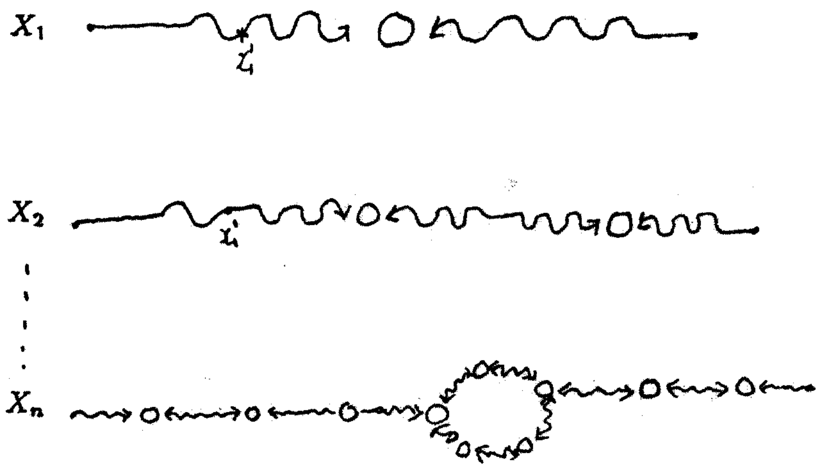
EXAMPLE 3.6. *Construct a nondegenerate continuum X in $\mathbb{R}^2$ such that each nondegenerate subcontinuum of X separates $\mathbb{R}^2$.*

Proof. Construct H as the below diagram ;



H

By the same way 3.3, do the replacing. Then



Hence each subset of $\lim_{i \to \infty} X_n$ contains many X_1. Therefore $\bigcirc$ in X_1 separates $\mathbb{R}^2$.

THEOREM 3.7. *Let* $X = \varprojlim\{X_i, f_i\}_{i=1}^{\infty}$ *where each* X_i *is a nonempty compact metric space with metric* d_i. *Let*

$$f_{ij} = f_i \circ \cdots \circ f_{j-1} : X_j \to X_i \text{ if } j > i+1 \text{ and } f_{i\,i+1} = f_i.$$

Let (Y, ρ) *be a complete metric space. Then, there is a sequence* $\{\epsilon_i\}_{i=1}^{\infty}$, $\epsilon_i > 0$, *such that if there are embeding* h_i *of* X_i *in* Y *satisfying* $\rho(h_j(x), h_j \circ f_{ij}(x)) < \frac{\delta_i}{3}$ *for each* x *in* X_i *and* $j > i$ *where* $\delta_i < \frac{1}{i}$ *and* $\delta < \rho(h_j(x), h_j(x))$ *whenever* $d_i(y, z) > \epsilon_i$, *then* X *is embedable in* Y.

Proof. Let d be a metric for X defined by, for each $(x_i)_{i=1}^{\infty}$, $(y_i)_{i=1}^{\infty}$ in X,

$$d((x_i)_{i=1}^{\infty}, (y_i)_{i=1}^{\infty}) = \sum_{i=1}^{\infty} 2^{-i} \frac{d_i(x_i, y_i)}{1 + d_i(x_i, y_i)}.$$

Then i-th projection $\pi_i : X \to X_i$ is 2^{-1}-map ; for all $x = (x_1, \cdots, x_i, x_{i+1}, \cdots)$, $y = (x_1, x_2, \cdots, x_i, y_{i+1}, y_{i+2} \cdots)$ in $\pi_i^{-1}(x)$. Then

$$\begin{aligned} d(x,y) &= 2^{-(i+1)}\left(\frac{d_{i+1}(x_{i+1}, y_{i+1})}{1 + d_{i+1}(x_{i+1}, y_{i+1})}\right) \\ &\quad + 2^{-(i+2)}\left(\frac{d_{i+2}(x_{i+2}, y_{i+2})}{1 + d_{i+2}(x_{i+2}, y_{i+2})}\right) \\ &\quad \vdots \\ &= 2^{-i}\left(2^{-1}\left(\frac{d_{i+1}(x_{i+1}, y_{i+1})}{1 + d_{i+1}(x_{i+1}, y_{i+1})}\right) + \cdots\right) < 2^{-i}. \end{aligned}$$

Make $\epsilon_i(\subset X_i) = \{y \in X_i | \pi_i^{-1}(y) \in \pi_i^{-1}(x_i)$ where x_i is fixed in $X_i\}$. Then $\{\epsilon_i\}_{i=1}^{\infty}$ converges to one point. Let h be $\lim_{i\to\infty}\{h_i \circ \pi_i(x)\}$. Since Y is a complete metric space, $\{h_i \circ \pi_i(x)\}$ is converges. So $\{h_i\}$ is a Cauchy sequence. h is one to one ; $x, y \in X$ such that $x = (x_1, x_2, \cdots), y = (y_1, y_2, \cdots)$. Assume $h(x) \neq h(y)$. Then $\lim_{i\to\infty}\{h_i \circ \pi_i(x)\} \neq \lim_{i\to\infty}\{h_i \circ \pi_i(y)\}$. Since h_i is an embeding, $\pi_i(x) \neq \pi_i(y)$ and $x_i \neq y_i$ in X_i, since π_i is a 2^{-1}-map. Then $x_i \neq y_i, i = i, i+1, \cdots$. Hence h is one to one. h is onto ; trivial. h is continuous ; since π_i is continuous, $\lim_{i\to\infty} \pi_i$ is continuous and $\lim_{i\to\infty} h_i$ is continuous, h^{-1} is continuous ; define

$$O_y = \{x \in X | d(x, y) < 2^{-i}\}$$

and

$$U_y = \{x_i \in X_i | d_i(x_i, y_i) < 2^{-i}\}.$$

Then O_y, U_y are open and $\pi_i(O_y) \supset U_y \ni x_i$ and so $\pi_i(O_y)$ is open. Hence h^{-1} is continuous. Therefore X is embedable in Y.

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