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REAL HYPERSURFACES OF TYPE A OF A COMPLEX SPACE FORM

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0. Introduction

A Kaehlerian manifold of constant holomorphic sectional curvature c is called a complex space form, which is denoted by $M_n(c)$. A complete and simply connected complex space form consists of a complex projective space P_nC, a complex Euclidean space C_n or a complex hyperbolic H_nC, according as $c > 0, c = 0$ or $c < 0$.

Let M be a real hypersurface of an n-dimensional complex space form $M_n(c)$. Then M induces an almost contact metric structure (ϕ, ξ, g) from the structure of ambient space. We denote by A the shape operator in the direction of the unit normal C on M. Typical examples of real hypersurfaces in P_nC are homogeneous ones . Takagi [14] classified homogeneouse real hypersurfaces as six types of a complex projective space P_nC. This result is generalized by many authors ([2], [5], [6], [7], [12], etc.). In particular, Okumura [10] proved the following

THEOREM A. *If the shape operator A and the structure tensor ϕ commute to each other, then the real hypersurface of a complex projective space P_nC is locally a tube of radius r over one of the following :*
(A_1) *a hyperplane* $P_{n-1}C$, *where* $0 < r < \frac{\pi}{2}$,
(A_2) *a totally geodesic* $P_kC(1 \leq k \leq n-2)$, *where* $0 < r < \frac{\pi}{2}$.

On the other hand, real hypersurfaces of H_nC have been also investigated by many authors ([1], [4], [8] etc.) from the different points of view. Montiel and Romero [9] proved the following

THEOREM B. *Let M be a real hypersurface of $H_nC, n \geq 2$. If it satisfies $A\phi = \phi A$, then M is locally congruent to one of the following:*
(A_0) *a horosphere,*
(A_1) *a tube over a hyperplane* $H_{n-1}C$,

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(A_2) *a tube over a totally geodesic* $H_kC(1 \leq k \leq n-1)$.

Let M be a real hypersurface of type A_1 or type A_2 in a complex projective P_nC or that of type A_0, A_1 or A_2 in a complex hyperbolic H_nC. Such real hypersurfaces in Theorem A and Theorem B are said to be of type A. For the real hypersurface of type A the following theorem is provided in [3], [6].

THEOREM C. *Let M be a real hypersurface of a complex space form $M_n(c), c \neq 0, n \geq 2$. If it satisfies*

$$\nabla_\xi A = 0, \quad g(A\xi, \xi) \neq 0,$$

then M is locally of type A, where ∇ is the Riemannian connection on M.

Pyo [11] showed that the following generalized property of Theorem C.

THEOREM D. *Let M be a real hypersurface of $M_n(c), c \neq 0, n \geq 2$. If it satisfies*

$$\nabla_\xi A = a(A\phi - \phi A), \quad 2a \neq -g(A\xi, \xi)$$

for some nonzero constant a, then M is locally of type A.

The purpose of this article is to improve Theorem D, and give a simple computaional proof of this. Namely we prove

THEOREM 1. *Let M be a real hypersurface of a complex space form $M_n(c), c \neq 0, n \geq 2$. If it satisfies*

$$\nabla_\xi A = f(A\phi - \phi A), \quad g(A\xi, \xi) \neq -2f$$

for some function f, then M is locally of type A provided that $g(A\xi, \xi)$ is constant.

1. Preliminaries

Let $M_n(c)$ be a real $2n$-dimensional complex space form equipped with parallel almost complex structure J and a Riemannian metric tensor G which is J-Hermitian, and covered by a system of coordinate neighborhoods $\{W; x^A\}$.

Let M be a real $(2n-1)$-dimensional hypersurface of $M_n(c)$ covered by a system of coordinate neighborhoods $\{V; y^h\}$ and immersed isometrically in $M_n(c)$ by the immersion $i : M \to M_n(c)$. Throughout the present paper the following convention on the range of indices are used, unless otherwise stated:

$$A, B, \ldots = 1, 2, \ldots, 2n : i, j, \ldots = 1, 2, \ldots, 2n-1.$$

The summation convention will be used with respect to those system of indices. When the argument is local, M need not be distinguished from $i(M)$. Thus, for simplicity, a point p in M may be identified with the point $i(p)$ and a tangent vector X at p may also be identified with the tangent vector $i_*(X)$ at $i(p)$ via the differential i_* of i. We represent the immersion i locally by $x^A = x^A(y^h)$ and $B_j = (B_j{}^A)$ are also $(2n-1)$-linearly independent local tangent vectors of M, where $B_j{}^A = \partial_j x^A$ and $\partial_j = \partial/\partial y^j$. A unit normal C to M may then be chosen. The induced Riemannian metric g with components g_{ji} on M is given by $g_{ji} = G_{BA}B_j{}^B B_j{}^A$ because the immersion is isometric.

For the unit normal C to M, the following representations are obtained in each coordinate neighborhoods:

$$JB_i = \phi_i{}^h B_h + \xi_i C, \quad JC = -\xi^i B_i, \tag{1.1}$$

where we have put $\phi_{ji} = G(JB_j, B_i)$ and $\xi_i = G(JB_i, C), \xi^h$ being components of a vector field ξ associated with ξ_i and $\phi_{ji} = \phi_j{}^r g_{ri}$. By the properties of the almost Hermitian structure J, it is clear that ϕ_{ji} is skew-symmetric. A tensor field of type (1,1) with components $\phi_i{}^h$ will be denoted by ϕ. By the properties of the almost complex structure J, the following relations are then given:

$$\phi_i{}^r \phi_r{}^h = -\delta_i{}^h + \xi_i \xi^h, \quad \xi^r \phi_r{}^h = 0, \quad \xi_r \phi_i{}^r = 0, \quad \xi_i \xi^i = 1,$$

that is, the aggregate (ϕ, g, ξ) defines an almost contact metric structure.

Denoting by ∇_j the operator of van der Waerden-Bortolotti covariant differentiation with respect to the induced Riemannian metric, equations of the Gauss and Weingarten for M are respectively obtained:

$$\nabla_j B_i = A_{ji} C, \quad \nabla_j C = -A_j{}^r B_r, \tag{1.2}$$

where $H = (A_{ji})$ is a second fundamental form and $A = (A_j{}^h)$, which is related by $A_{ji} = A_j{}^r g_{ri}$ is the shape operator derived from C. By means of (1.1) and (1.2) the covariant derivatives of the structure tensors are yielded:

$$\nabla_j \phi_i{}^h = -A_{ji}\xi^h + A_j{}^h \xi_i, \quad \nabla_j \xi_i = -A_{jr}\phi_i{}^r. \tag{1.3}$$

Since the ambient space is complex space form, equations of the Gauss and Codazzi for M are respectively given by

$$(1.4)\qquad \begin{aligned} R_{kjih} =& \frac{c}{4}(g_{kh}g_{ji} - g_{jh}g_{ki} + \phi_{kh}\phi_{ji} - \phi_{jh}\phi_{ki} - 2\phi_{kj}\phi_{ih}) \\ &+A_{kh}A_{ji} - A_{jh}A_{ki}, \end{aligned}$$

$$(1.5)\qquad \nabla_k A_{ji} - \nabla_j A_{ki} = \frac{c}{4}(\xi_k\phi_{ji} - \xi_j\phi_{ki} - 2\xi_i\phi_{kj}),$$

where R_{kjih} are components of the Riemannian curvature tensor R of M. Applying ϕ^{ki} to (1.5), we obtain

$$(1.6)\qquad (\nabla_k A_{ji})\phi^{ki} = -\frac{c}{2}(n-1)\xi_j.$$

In what follows, to write our formulas in convention forms, we denote by $A^2_{ji} = A_{jr}A_i{}^r$, $h_2 = A_{ji}A^{ji}$, $h = g_{ji}A^{ji}$, $\alpha = A_{ji}\xi^j\xi^i$ and $\beta = A_{ji}{}^2\xi^j\xi^i$. If we prt $U_j = \xi^r\nabla_r\xi_j$, then U is orthogonal to the structure vector ξ. Because of the properties of the almost contact metric struture and the second equation of (1.3), we can get

$$(1.7)\qquad \phi_{jr}U^r = A_{jr}\xi^r - \alpha\xi_j,$$

which shows that $g(U,U) = \beta - \alpha^2$.

From (1.3), we have

$$\nabla_k\nabla_j\xi_i = (A_{jr}\xi^r)A_{ki} - A_{jk}{}^2\xi_i - (\nabla_k A_{jr})\phi_i{}^r,$$

with which together (1.6) implies that

$$\nabla_i\nabla_j\xi^i = hA_{jr}\xi^r - A_{jr}{}^2\xi^r + \frac{c}{2}(n-1)\xi_j.$$

Since we have $divU = (\nabla_j\xi_i)(\nabla^i\xi^j) + \xi^j\nabla_i\nabla_j\xi^i$, the above equation implies

$$||\nabla_j\xi_i + \nabla_i\xi_j||^2 = 2dviU + 2\{h_2 - \alpha h - \frac{c}{2}(n-1)\},$$

where $||X||^2 = g(X,X)$ for any vector field X on M. Thus we have

$$(1.8)\qquad divU = \frac{1}{2}||A\phi - \phi A||^2 - h_2 + \alpha h + \frac{c}{2}(n-1).$$

By the definition of U and the second equation of (1.3), we esaily see that

$$U^r\nabla_j\xi_r = A_{jr}{}^2\xi^r - \alpha A_{jr}\xi^r. \tag{1.9}$$

On the other hand, differentiating (1.7) covariantly along M and making use of (1.3), we find

$$\xi_j A_{kr}U^r + \phi_{jr}\nabla_k U^r = \xi^r\nabla_k A_{jr} - A_{jr}A_{ks}\phi^{rs} - \alpha_k\xi_j + \alpha A_{kr}\phi_j{}^r, \tag{1.10}$$

which shows that

$$(\nabla_k A_{ji})\xi^j\xi^i = 2A_{kr}U^r + \alpha_k. \tag{1.11}$$

2. Formulas

Let M be a real hypersurface of a complex space form satisfying $\nabla_\xi A = f(A\phi - \phi A)$ for some differentiable function f. Then we have

$$\xi^r\nabla_r A_{ji} = f(A_{jr}\phi_i{}^r + A_{ir}\phi_j{}^r). \tag{2.1}$$

From this and (1.11) we see that

$$2A_{jr}U^r + \alpha_j + fU_j = 0. \tag{2.2}$$

Thus, it follows that we have

$$\alpha_t\xi^t = 0. \tag{2.3}$$

because U is orthogonal to the structure vector ξ. Using the Codazzi equation (1.5), the relationship (2.1) turns out to be

$$\xi^r\nabla_j A_{ir} = -\frac{c}{4}\phi_{ji} + f(A_{jr}\phi_i{}^r + A_{ir}\phi_j{}^r), \tag{2.4}$$

which together with (1.10) implies that

$$\begin{aligned}-\frac{c}{4}\phi_{ji}+&f(A_{jr}\phi_i{}^r + A_{ir}\phi_j{}^r)\\ &=\phi_i{}^r\nabla_j U_r - A_{ir}\nabla_j\xi^r - \alpha A_{jr}\phi_i{}^r + (A_{jr}U^r + \alpha_j)\xi_i.\end{aligned}$$

If we transform by $\phi_k{}^i$, then we obtain

$$\frac{c}{4}(g_{jk} - \xi_j\xi_k) + fA_{jk} - f\xi_k A_{jr}\xi^r - fA_{sr}\phi_j{}^r\phi_k{}^s$$
$$= \nabla_j U_k + \xi_k U^r \nabla_j \xi_r - (\nabla_r \xi_k)(\nabla_j \xi^r) - \alpha A_{jk} + \alpha \xi_k A_{jr}\xi^r,$$

or take account of (1.9)

$$\begin{aligned} \nabla_j U_i = {} & (\nabla_j \xi^r)(\nabla_r \xi_i) - \xi_i(A_{jr}{}^2\xi^r + fA_{jr}\xi^r) \\ & + (f+\alpha)A_{ji} + \frac{c}{4}(g_{ji} - \xi_j\xi_i) - fA_{rs}\phi_j{}^r\phi_i{}^s. \end{aligned} \tag{2.5}$$

On the other hand, from (1.11) and (2.2) we have

$$\xi^r\xi^s\nabla_j A_{rs} = -fU_j.$$

Differentiating this covariantly along M and using (1.3) and (2.4), we get

$$(\nabla_k\nabla_j A_{sr})\xi^r\xi^s - \frac{c}{2}A_{jk} + \frac{c}{2}\xi_j A_{kr}\xi^r$$
$$= 2f(\nabla_k\xi_r)(\nabla_j\xi^r) + 2f(\nabla_k\xi^r)(\nabla_r\xi_j) - f_k U_j - f\nabla_k U_j,$$

where $f_i = \partial_i f$. If we take account of the Ricci formula for A_{rs}, then we have

$$-2R_{kjih}\xi^i A_t{}^h\xi^t + \frac{c}{2}(\xi_j A_{kr}\xi^r - \xi_k A_{jr}\xi^r)$$
$$= 2f\{(\nabla_k\xi^r)(\nabla_r\xi_j) - (\nabla_j\xi^r)(\nabla_r\xi_k)\} + f_j U_k - f_k U_j + f(\nabla_j U_k - \nabla_k U_j).$$

Thus, we verify, making use of (1.4), that

$$2\{(A_{kr}\xi^r)(A_{js}{}^2\xi^s) - (A_{jr}\xi^r)(A_{ks}{}^2\xi^s)\}$$
$$= 2f\{(\nabla_k\xi^r)(\nabla_r\xi_j) - (\nabla_j\xi^r)(\nabla_r\xi_k)\} + f_j U_k - f_k U_j + f(\nabla_j U_k - \nabla_k U_j),$$

which together with (2.5) yields

$$\begin{aligned} & f\{(\nabla_j\xi^r)(\nabla_r\xi_i) - (\nabla_i\xi^r)(\nabla_r\xi_j)\} + f_i U_j - f_j U_i \\ & = A_{jr}\xi^r\{2A_{is}{}^2\xi^s - f^2\xi_i\} - A_{ir}\xi^r\{2A_{js}{}^2\xi^s - f^2\xi_j\} \\ & + f(\xi_j A_{ir}{}^2\xi^r - \xi_i A_{jr}{}^2\xi^r). \end{aligned} \tag{2.6}$$

Transvecting (2.6) with ξ^j and using (1.3), we obtain
(2.7)

$$-fU^rA_{rt}\phi_i{}^t = (\xi^t f_t)U_i + \alpha(2A_{is}{}^2\xi^s - f^2\xi_i) - (2\beta - f^2)A_{ir}\xi^r + f(A_{ir}{}^2\xi^r - \beta\xi_i).$$

If we transvect U^i to this and make use of (1.7), we get

$$(\beta - \alpha^2)f_t\xi^t = 0.$$

Let M_0 be the non-empty open subset of M such that $\beta(x) - \alpha^2(x) \neq 0, x \in M$. Then we have

$$f_t\xi^t = 0 \quad \text{on } M_0.$$

Here after unless otherwise stated, our discussion will be continued on M_0. Thus (2.7) is reduced to

$$(2.8) \quad -fU^rA_{rs}\phi_i{}^s = (f + 2\alpha)\xi^r A_{ir}{}^2 + (f^2 - 2\beta)A_{ir}\xi^r - f(\beta + f\alpha)\xi_i.$$

Hence we have

$$(2.9) \qquad fA_{jr}U^r = (f + 2\alpha)\xi^r A_{rs}{}^2\phi_j{}^s + (2\beta - f^2)U_j.$$

Substituting (2.2) into (2.8), we obtain

$$(2.10) \qquad 2(f + 2\alpha)\xi^r A_{rs}{}^2\phi_j{}^s + (4\beta - f^2)U_j + f\alpha_j = 0.$$

Transforming (2.6) by $\phi_k{}^i$ and using the second equation of (1.3), we have

$$\begin{aligned} f(A_{js}A_{kr}\phi^{rs} + T_{sr}\phi_j{}^s\phi_k{}^r) \\ = U_jf_r\phi_k{}^r - f_j(A_{kr}\xi^r - \alpha\xi_k) - U_k(2A_{jr}{}^2\xi^r - f^2\xi_j) \\ + f\xi_kA_{jr}U^r - (2A_{jr}\xi^r + f\xi_j)\xi^sA_{st}{}^2\phi_k{}^t, \end{aligned}$$

where we have put $T_{ji} = A_{js}A_{ir}\phi^{rs}$. Because the left hand side of (2.10) is skew-symmetric with respect to j and k, it follows that we obtain
(2.11)

$$\begin{aligned} &U_jf^r\phi_{kr} + U_kf^r\phi_{jr} - f_j(A_{kr}\xi^r - \alpha\xi_k) - f_k(A_{jr}\xi^r - \alpha\xi_j) \\ &+f(\xi_jA_{kr}U^r + \xi_kA_{jr}U^r) - U_k(2A_{jr}{}^2\xi^r - f^2\xi_j) - U_j(2A_{kr}{}^2\xi_r - f^2\xi_k) \\ &-(2A_{jr}\xi^r + f\xi_j)\xi^sA_{st}{}^2\phi_k{}^t - (2A_{kr}\xi^r + f\xi_k)\xi^sA_{st}{}^2\phi_j{}^t = 0. \end{aligned}$$

Transvecting the last equation with $U_s\phi^{js}$ and making use of (1.3) and (1.7), we get

$$(2.12)\qquad \begin{aligned}&(U_rf^r - 2A_{tr}{}^2\xi^rU_s\phi^{ts})U_i - \phi_{js}U^sf^j(A_{ir}\xi_r - \alpha\xi_i)\\ &-(\beta-\alpha^2)\{f_i + 2\xi^rA_{sr}{}^2\phi_i{}^s\} + f(A_{jr}U^rU_s\phi^{js})\xi_i = 0,\end{aligned}$$

which together with (2.8) give $fA_{js}U^rU_s\phi^{js} = 0$ and hence $(\beta-\alpha^2)\phi_{js}U^sf^j = 0$. Therefore (2.12) turns out to be

$$(2.13)\qquad (\beta-\alpha^2)(f_i + 2\xi^rA_{sr}{}^2\phi_i{}^s) = (U_rf^r + 2U_sA_{tr}{}^2\xi^r\phi^{st})U_i.$$

Thus, (2.10) is reduced to

$$(2.14)\qquad f\alpha_j = (f+2\alpha)f_j - \lambda U_j,$$

where we have defined

$$(2.15)\qquad \lambda = \frac{f+2\alpha}{\beta-\alpha^2}(U_tf^t + 2U_sA_{tr}{}^2\xi^r\phi^{st}) + 4\beta - f^2.$$

PROPOSITION 2.1. *Let M be a real hypersurface of a complex projective space satisfying (2.1). If f is constant, then ξ is principal.*

Remark. In the proof of Theorem C [3], it is seen that $f \neq 0$ on M_0. We will give a simple proof of Proposition 3.1 of [11].

Proof. Since $f_j = 0$, it is seen, using (2.6), that

$$(2.16)\qquad A_{jr}{}^2\xi^r = \mu A_{jr}\xi^r,$$

where we have put $\alpha\mu = \beta$, for detail see Lemma 3.2 of [11]. Thus (2.9) turns out to be

$$(2.17)\qquad A_{jr}U^r = -(\mu+f)U_j.$$

Differentiating (2.16) covariantly and making use of (1.3), we find

$$(2.18)\qquad \begin{aligned}&(\nabla_kA_{js})A_r{}^s\xi^r + A_j{}^i(\nabla_kA_{ir})\xi^r - \mu(\nabla_kA_{jr})\xi^r\\ &=\mu_kA_{jr}\xi^r + A_{jr}{}^2A_{ks}\phi^{rs} - \mu A_{kr}A_{js}\phi^{rs}.\end{aligned}$$

If we transvect ξ^j to this and make use of (2.1), (2.16) and (2.17), then we obtain

$$\alpha\mu_k = (\frac{c}{2} - 2f^2 + 2\mu^2 - \mu f)U_k. \tag{2.19}$$

Substituting (2.1) into (2.18), then we obtain

$$\begin{aligned}(\nabla_k A_{js})A_{rs}\xi^r + &\frac{c}{4}(\mu\phi_{kj} - A_{jr}\phi_k{}^r) + (f-\mu)A_{js}A_{kr}\phi^{sr}\\ &+ fA_{jr}{}^2\phi_k{}^r - \mu f(A_{kr}\phi_j{}^r + A_{jr}\phi_k{}^r)\\ &= \mu_k A_{jr}\xi^r + A_{jr}{}^2 A_{ks}\xi^{rs},\end{aligned}$$

from which, taking the skew-symmetric part,

(2.20)

$$\begin{aligned}\frac{c}{4}(U_k\xi_j - U_j\xi_k) + \frac{c}{2}(\mu-\alpha)\phi_{kj} - &\frac{c}{4}(A_{jr}\phi_k{}^r - A_{kr}\phi_j{}^r)\\ &+ 2(f-\mu)A_{js}A_{kr}\phi^{sr} + f(A_{jr}{}^2\phi_k{}^r - A_{kr}{}^2\phi_j{}^r)\\ &= \mu_k A_{jr}\xi^r - \mu_j A_{kr}\xi^r + A_{jr}{}^2 A_{ks}\phi^{rs} - A_{kr}{}^2 A_{js}\phi^{rs}.\end{aligned}$$

Transforming (2.20) with U^j and using (1.7), (2.16), (2.17) and (2.19), we obtain

$$\begin{aligned}&-\frac{c}{4}\alpha(\mu-\alpha)\xi_k + \frac{c}{2}(\mu-\alpha)(A_{kr}\xi^r - \alpha\xi_k) + \frac{c}{4}(\mu+f)(A_{kr}\xi^r - \alpha\xi_k)\\ &-\frac{c}{4}(\mu-\alpha)A_{kr}\xi^r + 3(f^2+\mu f)(\mu-\alpha)A_{kr}\xi^r + f(\mu+f)^2(A_{kr}\xi^r - \alpha\xi_k)\\ &= -(\mu-\alpha)(\frac{c}{2} - 2f^2 + 2\mu^2 - \mu f)A_{kr}\xi^r + \mu^2(\mu-\alpha)A_{kr}\xi^r,\end{aligned}$$

which implies $A_{jr}\xi^r = \alpha\xi_j$. This completes the proof.

3. Proof of Theorem 1

Let M be a real hypersurface of $M_n(c)$. $c \neq 0$ satisfying (2.1). If α is constant, then (2.2) and (2.14) are reduced respectively to

$$2A_{jr}U^r = -fU_j, \tag{3.1}$$

$$(f+2\alpha)f_j = \lambda U_j. \tag{3.2}$$

Thus, we have $\lambda \neq 0$ on M_0. Differentiating (3.2) covariantly along M, we find

$$(f_k + 2\alpha_k)f_j + (f + 2\alpha)\nabla_k f_j = \lambda_k U_j + \lambda \nabla_k U_j,$$

from which,

$$\lambda_k U_j - \lambda_j U_k + \lambda(\nabla_k U_j - \nabla_j U_k) = 0.$$

If we transvect this by U^j and take account of (2.5), (2.6) and (3.1), then we obtain $(\beta - \alpha^2)\lambda_k = (\lambda_t U^t)U_k$. Thus, it follows that we have on M_0

$$\nabla_j U_i - \nabla_i U_j = 0 \tag{3.3}$$

because of $\lambda \neq 0$ on M_0.

Transvecting (3.3) with U^j and making use of (1.9) and (2.5), we find

$$A_{jr}{}^2\xi^r + fA_{jr}\xi^r - A_{sr}U^r\phi_j{}^s - (\beta + \alpha f)\xi_j = 0,$$

which together with (3.1) implies that

$$A_{jr}{}^2\xi^r = -\frac{3}{2}fA_{jr}\xi^r + (\beta + \frac{3}{2}\alpha f)\xi_j.$$

From this and (2.10), it is seen that

$$2\beta + 3\alpha f + f^2 = 0. \tag{3.4}$$

Thus, it follows that we have

$$2A_{jr}{}^2\xi^r + 3fA_{jr}\xi^r + f^2\xi_j = 0 \tag{3.5}$$

and consequently

$$A_{jr}{}^2\xi^r - \alpha A_{jr}\xi^r = -\frac{1}{2}(2\alpha + 3f)A_{jr}\xi^r - \frac{1}{2}f^2\xi_j, \tag{3.6}$$

$$A_{jr}{}^3\xi^r - \alpha A_{jr}{}^2\xi^r = \frac{1}{4}f(6\alpha + 7f)A_{jr}\xi^r + \frac{1}{4}f^2(2\alpha + 3f)\xi_j. \tag{3.7}$$

Differentiating (3.5) covariantly and using (1.3), we find

$$\begin{aligned} &2(\nabla_k A_{js})A_r{}^s\xi^r + 2A_j{}^s(\nabla_k A_{rs})\xi^r - 2A_{jr}{}^2A_{ks}\phi^{rs} + 3f_k A_{jr}\xi^r \\ &+3f(\nabla_k A_{jr})\xi^r - 3fA_{jr}A_{ks}\phi^{rs} + 2ff_k\xi_j - f^2A_{kr}\phi_j{}^r = 0, \end{aligned} \tag{3.8}$$

or substituting (2.1)

$$2(\nabla_k A_{js})A_r{}^s\xi^r - \frac{c}{2}A_{jr}\phi_k{}^r - f^2 A_{kr}\phi_j{}^r - fA_{jr}A_{ks}\phi^{rs} + 2fA_{jr}{}^2\phi_k{}^r$$
$$-2A_{jr}{}^2A_{ks}\phi^{rs} + 3f(\nabla_k A_{jr})\xi^r + 3f_kA_{jr}\xi^r + 2ff_k\xi_j = 0.$$

If we take the skew-symmetric part of this and use (1.5), then we obtain (3.9)

$$\frac{c}{2}(U_k\xi_j - U_j\xi_k) - \frac{c}{2}(\alpha + 3f)\phi_{kj} + (f^2 + \frac{c}{2})(A_{jr}\phi_k{}^r - A_{kr}\phi_j{}^r)$$
$$-2fA_{jr}A_{ks}\phi^{rs} + 2f(A_{jr}{}^2\phi_k{}^r - A_{kr}{}^2\phi_j{}^r) + 2(A_{kr}{}^2A_{js}\phi^{rs} - A_{jr}{}^2A_{ks}\phi^{rs})$$
$$+3(f_kA_{jr}\xi^r - f_jA_{kr}\xi^r) + 2f(f_k\xi_j - f_j\xi_k) = 0.$$

Transvecting (3.9) with U^k and making use of (1.7), (3.4), (3.5), (3.6) and (3.7), we find

$$\{3U_tf^t - (\alpha + f)f^2 + \frac{c}{2}(2\alpha + 5f)\}A_{jr}\xi^r$$
$$+\{2f(U_tf^t) - \frac{1}{2}(\alpha + f)f^3 - \frac{c}{2}\alpha(2\alpha + 5f)\}\xi_j = 0.$$

Thus, we have $3U_tf^t = (\alpha+f)f^2 - \frac{c}{2}(2\alpha+5f)$ on M_0 and hence $(\alpha+f)f^3 = c(2\alpha+5f)(2f+3\alpha)$. Since α is constant, it is clear that f is constant. By Proposition 2.1, M_0 is empty. Hence ξ is principal. Owing to Theorem C and Theorem D, the conclusion of Theorem 1 is true.

References

1. J. Berndt, *Real hypersurfaces with constant principal curvatures in a complex hyperbolic space*, J. Reine Angew. Math., **395** (1989), 132-141.
2. T. E. Cecil and P. J. Ryan, *Focal sets and real hypersurfaces in complex projective space*, Trans. Amer. Math Soc., **269** (1982), 481-499.
3. U-H. Ki, S-J. Kim and S-B. Lee, *Some characterizations of real hypersurfaces of type A*, Kyungpook Math. J., **31** (1991), 73-82.
4. U-H. Ki, Y. J Suh, *On real hypersurfaces of a complex space form*, Math. J. Okayama., **32** (1990), 207-221.
5. M. Kimira and S. Maeda, *Sectional curvatures of holomorphic planes on a real hypersurface in P_nC*, Math. ann., **276** (1987), 487-497.
6. M. Kimura and S. Maeda, *On real hypersurfaces of a complex projective space II*, Tsukuba J. Math., **15** (1991), 547-561.
7. S. Maeda and S. Udagawa, *Real hypersurfaces of a complex projective space in terms of holomorphic distribution*, Tsukuba J. Math., **14** (1990), 39-52.
8. S. Montiel, *Real hypersurfaces of a complex hyperbolic space*, J. Math. Soc. Japan., **37** (1985), 515-535.

9. S. Montiel and A. Romero, *On some real hypersurfaces of a complex hyperbolic space*, Geometriae Dedocara., **20** (1986), 245-261.
10. M. Okumura, *On some real hypersurfaces of a complex projective space*, Trans. Amer. Math. Soc., **212** (1975), 355-364.
11. Y-S. Pyo, *On real hypersurfaces of type A in a complex space from I*, Tsukuba J. Math., **18** (1994), 483-492.
12. Y-S. Pyo, *On real hypersurfaces of type A in a complex space from II*, Comm. Korean Math. Soc., **9** (1994), 369-383.
13. R. Takagi, *On homogeneous real hypersurfaces in a complex projective space*, Osaka J. Math., **10** (1973), 495-506.
14. R. Takagi, *Real hypersurfaces in a complex projective space with constant principal curvature I, II*, J. Math. Soc. Japan., **27** (1975), 43-53, 507-516.