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SOME PROPERTIES OF THE HOMOTOPY SET OF THE AXES OF PAIRINGS

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1. Introduction

Varadarajan([4]) defined the generalized Gottlieb set $G(A,X)$ of the homotopy classes of the cyclic maps $f: A \to X$ and studied the fundamental properties of $G(A,X)$. If A is a co-Hopf space, then Varadarajan set $G(A,X)$ has a group structure. This group $G(A,X)$ is a generalization of $G(X)$ and $G_n(X)$ of Gottlieb([2],[3]). Some authors studied the properties of the Varadarajan set, its dual and related topics.

We write $f \perp g$ when there exists a continuous map $\mu: X \times Y \to Z$ with axes $f: X \to Z$ and $g: Y \to Z$.

Let $v: X \to Z$ be a fixed map. We define the set of the homotopy classes of the axes by

$$v^{\perp}(Y,Z) = \{[g]: Y \to Z \mid v \perp g\}.$$

This set depends only on the homotopy type of the spaces X, Y and Z and the homotopy class of v. If $X = Z$ and $v \simeq 1_X$, then $(1_X)^{\perp}(Y,X)$ is exactly the Varadarajan set $G(Y,X)$ ([4]).

The purpose of this paper is to find some properties on the set of the homotopy classes of the axes of pairings. We shall work in the category of spaces with base points and having the homotopy type of locally finite CW-complexes. All maps shall mean continuous functions. All homotopies and maps are to respect base points. The base points will be denoted by $*$. For simplicity, we use the same symbol for a map and homotopy class.

2. Some properties of the homotopy classes of the axes of pairings

The map $\triangle_X : X \to X \times X$ denotes the diagonal map and $\nabla_X : X \vee X \to X$ the folding map defined by $\nabla_X(x,*) = x = \nabla_X(*,x)$ for any element x of X.

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DEFINITION 2.1. We call a map $\mu\colon X \times Y \to Z$ a *pairing* with the axes $f\colon X \to Z$ and $g\colon Y \to Z$, when it satisfies

$$\mu \circ j \simeq \nabla_Z \circ (f \vee g)\colon X \vee Y \to Z,$$

where $j\colon X \vee Y \to X \times Y$ is the inclusion.

We write $f \perp g$ when there exists a pairing $\mu\colon X \times Y \to Z$ with axes $f\colon X \to Z$ and $g\colon Y \to Z$.

Given a pairing $\mu\colon X \times Y \to Z$, we define a map $\alpha \dot{+} \beta\colon B \to Z$ for any map $\alpha\colon B \to X$ and $\beta\colon B \to Y$ by

$$\alpha \dot{+} \beta = \mu \circ (\alpha \times \beta) \circ \Delta_B.$$

This defines a function $\dot{+}\colon [B, X] \times [B, Y] \to [B, Z]$.

As well known,we have the following properties:

PROPERTY A.

(1) *Let $f_1\colon X_1 \to Z, f_2\colon X_2 \to X_1, g_1\colon Y_1 \to Z$ and $g_2\colon Y_2 \to Y_1$ be maps. Then $f_1 \perp g_1$ implies $(f_1 \circ f_2) \perp (g_1 \circ g_2)$.*
(2) *Let $f\colon X \to Z, g\colon Y \to Z$ and $w\colon Z \to W$ be maps. Then $f \perp g$ implies $(w \circ f) \perp (w \circ g)$.*

PROPERTY B.

(1) *If $f\colon A \to X$ is a cyclic map, then $f \circ g\colon B \to X$ is a cyclic map for any map $g\colon B \to A$.*
(2) *Let $f\colon X \to Z$ and $g\colon Y \to Z$ be maps. If $f \perp 1_Z$, or if $1_Z \perp g$, then $f \perp g$.*

DEFINITION 2.2. *A map $\theta\colon A \to B \vee C$ is called a copairing with coaxes $h\colon A \to B$ and $r\colon A \to C$ if it satisfies the condition that*

$$j \circ \theta \simeq (h \times r) \circ \Delta_A\colon A \to B \times C$$

for the inclusion map $j\colon B \vee C \to B \times C$.

Given a copairing $\theta\colon A \to B \vee C$, we define a map $\alpha \dot{+} \beta\colon A \to X$ for any maps $\alpha\colon B \to X$ and $\beta\colon C \to X$ by $\alpha \dot{+} \beta = \nabla_X \circ (\alpha \vee \beta) \circ \theta$.

This defines a function $\dot{+}\colon [B, X] \times [C, X] \to [A, X]$.

Theorem 2.3. *Let $\mu\colon X \times Y \to Z$ be a pairing with axes $f\colon X \to Z$ and $g\colon Y \to Z$, and $\theta\colon A \to H \vee R$ a copairing with coaxes $h\colon A \to H$ and $r\colon A \to R$. Let $\alpha\colon H \to X, \beta\colon R \to X, \gamma\colon H \to Y$ and $\delta\colon R \to Y$ be maps. Then*

(1) $(\alpha \dotplus \beta) \underset{\cdot}{+} (\gamma \dotplus \delta) = (\alpha \underset{\cdot}{+} \gamma) \dotplus (\beta \underset{\cdot}{+} \delta)$

(2) $h^*(\alpha) \underset{\cdot}{+} r^*(\delta) = f_*(\alpha) \dotplus g_*(\delta)$

(3) $r^*(\beta) \underset{\cdot}{+} h^*(\gamma) = f_*(\gamma) \dotplus g_*(\beta)$

where $h^* : [H, X] \to [A, X]$ and $f_* : [H, X] \to [H, Z]$ are induced maps by h and f respectively.

Proof.

(1)

$$
\begin{aligned}
(\alpha \dotplus \beta) \underset{\cdot}{+} (\gamma \dotplus \delta) &= \mu\{(\nabla_X \circ (\alpha \vee \beta) \circ \theta) \times (\nabla_Y \circ (\gamma \vee \delta) \circ \theta)\}\Delta_A \\
&= \nabla_Z\{(\mu \circ (\alpha \times \gamma) \circ \Delta_H) \vee (\mu \circ (\beta \times \delta) \circ \Delta_R)\} \circ \theta \\
&= (\alpha \underset{\cdot}{+} \gamma) \dotplus (\beta \underset{\cdot}{+} \delta)
\end{aligned}
$$

(2) Take $\beta = \gamma = *$ in (1). Note that $\nabla_X \circ (\alpha \vee *) = \alpha \circ P_1 \circ j$, where $j\colon H \vee R \to H \times R$ is the inclusion and $P_1\colon H \times R \to H$ is the projection. Then

$$
\begin{aligned}
\alpha \dotplus * &= \nabla_X \circ (\alpha \vee *) \circ \theta \\
&= \alpha \circ P_1 \circ j \circ \theta = h^*(\alpha)
\end{aligned}
$$

Similarly $* \dotplus \delta = r^*(\delta)$. And note that $\alpha \underset{\cdot}{+} * = \mu \circ (\alpha \times *) \circ \Delta_H$ and so $\mu \circ j \simeq \nabla_Z \circ (f \vee g)$ implies $\alpha \underset{\cdot}{+} * \simeq f \circ \alpha$, i.e., $\alpha \underset{\cdot}{+} * = f_*(\alpha)$. Similarly, $* \underset{\cdot}{+} \delta = g_*(\delta)$.

Thus we obtain that

$$h^*(\alpha) \underset{\cdot}{+} r^*(\delta) = f_*(\alpha) \dotplus g_*(\delta)$$

(3) is similarly proved.

Proposition 2.4([1]). *Let $f\colon X \to Z, h\colon V \to Z, g\colon Y \to V$ and $w\colon W \to V$ be maps. Suppose that $f \perp h$ and $g \perp w$. Then the following results hold.*

(1) $\{\nabla_Z \circ (f \vee (h \circ g))\} \perp (h \circ w)$

(2) *If $\theta\colon A \to X \vee Y$ is a copairing, then $\{f \dotplus (h \circ g)\} \perp (h \circ w)$.*

Proof. (1) Let $\mu_1 \colon X \times V \to Z$ and $\mu_2 \colon Y \times W \to V$ be pairings for $f \perp h$ and $g \perp w$ respectively. Define $\mu = \mu_1 \circ (1_X \times \mu_2) \circ (j \times 1_W) \colon (X \vee Y) \times W \to Z$, where $j \colon X \vee Y \to X \times Y$ is the inclusion. Then μ is a pairing for $\{\nabla_Z \circ (f \vee (h \circ g))\} \perp (h \circ w)$.

(2) By (1) and Property A, this holds.

COROLLARY 2.5. *Let $v \colon X \to Z$ be a map with a right homotopy inverse and $g_1 \colon Y_1 \to Z$ and $g_2 \colon Y_2 \to Z$ be maps. Then*

(1) *$v \perp g_1$ and $v \perp g_2$ imply $v \perp \{\nabla_Z \circ (g_1 \vee g_2)\}$.*
(2) *Let $\theta \colon Y \to Y_1 \vee Y_2$ be a copairing. Then $v \perp g_1$ and $v \perp g_2$ imply $v \perp (g_1 \dotplus g_2)$.*

Proof. Let $v' \colon Z \to X$ be a map with $vv' \simeq 1$. Then $g_1 \perp v$ implies $g_1 \perp 1_Z$, so that we can apply the above proposition.

Let $v \colon X \to Z$ be a map with right homotopy inverse and $\theta \colon Y \to Y_1 \vee Y_2$ be a copairing. By above corollary, we obtain a function

$$\dotplus \colon v^{\perp}(Y_1, Z) \times v^{\perp}(Y_2, Z) \to v^{\perp}(Y, Z)$$

by $g_1 \dotplus g_2 = \nabla_Z \circ (g_1 \vee g_2) \circ \theta$.

If $\theta \colon Y \to Y \vee Y$ is a copairing, i.e., Y is a co-Hopf space, then θ defines a binary operation on $v^{\perp}(Y, Z)$.

THEOREM 2.6. *Let $v \colon X \to Z$ be a map with a right homotopy inverse and let $w \colon Z \to W$ and $a \colon A \to Y$ be homotopy equivalences, then the following results hold.*

(1) *If Y is a co-Hopf space, then $w_* \colon v^{\perp}(Y, Z) \to (w \circ v)^{\perp}(Y, W)$ is an isomorphism.*
(2) *If A, Y are co-Hopf spaces and a is a co-Hopf map, then $a^* \colon v^{\perp}(Y, Z) \to v^{\perp}(A, Z)$ is an isomorphism.*

Proof. (1) Since Y is a co-Hopf space, $v^{\perp}(Y, Z)$ has a binary operation $\dotplus$. For $\alpha \in v^{\perp}(Y, Z)$, $w_*(\alpha) = w \circ \alpha$ and $w \circ \alpha \perp w \circ v$, so that $w_*(\alpha) \in (w \circ v)^{\perp}(Y, W)$. If $\alpha, \beta \in v^{\perp}(Y, Z)$, then $w_*(\alpha \dotplus \beta) = w \circ \{\nabla_Z \circ (\alpha \vee \beta) \circ \theta\} = \nabla_W \circ (w \circ \alpha \vee w \circ \beta) \circ \theta = w \circ \alpha \dotplus w \circ \beta = w_*(\alpha) \dotplus w_*(\beta)$. Thus w_* is a homomorphism. And it can be easily checked that w_* is a bijection.

(2) For $\alpha \in v^{\perp}(Y, Z), a^*(\alpha) = \alpha \circ a$, and $\alpha \circ a \perp v$, i.e., $a^*(\alpha) \in v^{\perp}(A, Z)$, so that a^* is well-defined. Let $\theta_1 \colon A \to A \vee A$ and $\theta_2 \colon Y \to Y \vee Y$ be copairings. Since $a \colon A \to Y$ is a co-Hopf map, $(a \vee a) \circ \theta_1 \simeq \theta_2 \circ a$. For

$\alpha, \beta \in v^{\perp}(Y, Z)$,

$$\begin{aligned} a^*(\alpha \dot{+} \beta) &= \nabla_Z \circ (\alpha \vee \beta) \circ \theta_2 \circ a \\ &= \nabla_Z \circ (\alpha \vee \beta) \circ (a \vee a) \circ \theta_1 \\ &= \nabla_Z \circ (\alpha \circ a \vee \beta \circ a) \circ \theta_1 \\ &= a^*(\alpha) \dot{+} a^*(\beta) \end{aligned}$$

Thus a^* is a homomorphism. The bijection of a^* comes from the fact that a is a homotopy equivalence.

3. The Dual Results of §2.

Now we study the duals of the previous results. We write $h \top r$ if there exists a copairing $\theta\colon A \to B \vee C$ with coaxes $h\colon A \to B$ and $r\colon A \to C$. Let $u\colon A \to B$ be a fixed map. We call a map $r\colon A \to C$ u-cocyclic if $u \top r$. We can now define the following set of the homotopy classes of the u-cocyclic maps, $r\colon A \to C$;

$$u^{\top}(A, C) = \{[r]\colon A \to C | u \top r\}$$

If $u \simeq 1_A$, then $(1_A)^{\top}(A, C)$ is just a Varadarajan's $DG(A, C)$.

PROPOSITION 3.1.

(1) *Let* $h_1\colon A \to B_1, h_2\colon B_1 \to B_2, r_1\colon A \to C_1$ *and* $r_2\colon C_1 \to C_2$ *be maps. Then* $h_1 \top r_1$ *implies* $(h_2 \circ h_1) \top (r_2 \circ r_1)$.

(2) *Let* $h\colon A \to B, r\colon A \to C$ *and* $d\colon D \to A$ *be maps. Then* $h \top r$ *implies* $(h \circ d) \top (r \circ d)$.

PROPOSITION 3.2. *Let* $h\colon A \to B, r\colon A \to C, u\colon B \to U, d\colon B \to D$ *be maps. Suppose that* $u \top d$ *and* $h \top \gamma$. *Then the following results hold.*

(1) $(u \circ h) \top \{((d \circ h) \times r) \circ \Delta_A\}$.

(2) *If* $\mu\colon D \times C \to Z$ *is a pairing, then*

$$(u \circ h) \top \{(d \circ h) \dot{+} r\}.$$

Proof. (1) If $\theta_1\colon A \to B \vee C$ and $\theta_2\colon B \to U \vee D$ are copairings for $h \top r$ and $u \top d$ respectively, then

$\theta = (1_U \vee j) \circ (\theta_2 \vee 1_C) \circ \theta_1\colon A \to B \vee C \to U \vee D \vee C \to U \vee (D \times C)$,

where $j\colon D \vee C \to D \times C$ is the inclusion map, is a copairing for $(u \circ h) \top \{((d \circ h) \times r) \circ \Delta_A\}$.

(2) By (1) and proposition 3.1.(1),

$(u \circ h) \top \{\mu \circ ((d \circ h) \times r) \circ \Delta_A\}$, i.e., $(u \circ h) \top ((d \circ h) \dot{+} r)$.

THEOREM 3.3. *Let $u\colon A \to B$ be a map with a left homotopy inverse and let $g_1\colon A \to C_1$ and $g_2\colon A \to C_2$ be maps. Then*

(1) *If $u \top g_1$ and $u \top g_2$, then $u \top \{(g_1 \times g_2) \circ \Delta_A\}$.*
(2) *If $\mu\colon C_1 \times C_2 \to Z$ is a pairing, then $u \top g_1$ and $u \top g_2$ imply $u \top (g_1 \dot{+} g_2)$.*

proof. Since u has a left homotopy inverse, say, u', $g_2 \top u$ implies $g_2 \top u'u$, i.e., $g_2 \top 1_A$. We apply the above proposition.

If $u\colon A \to B$ is a map with left homotopy inverse, and if $\mu\colon C_1 \times C_2 \to C$ is a pairing, then we obtain an induced function

$$\dot{+}\colon u^{\top}(A, C_1) \times u^{\top}(A, C_2) \to u^{\top}(A, C)$$

by $g_1 \dot{+} g_2 = \mu \circ (g_1 \times g_2) \circ \Delta_A$.

THEOREM 3.4. *Let $u\colon A \to B$ be a map with a left homotopy inverse and let $a\colon V \to A$ and $d\colon C \to D$ be homotopy equivalences. Then the following results hold.*

(1) *$a^*\colon u^{\top}(A, C) \to (u \circ a)^{\top}(V, C)$ is an isomorphism.*
(2) *If C and D are Hopf spaces and $d\colon C \to D$ is a Hopf map, then*

$$d^*\colon u^{\top}(A, C) \to u^{\top}(A, D)$$

is an isomorphism.

Proof. (1) For $\alpha \in u^{\top}(A, C), a^*(\alpha) = \alpha \circ a$ and $\alpha \circ a \top u \circ a$, so that $a^*(\alpha) \in (u \circ a)^{\top}(V, C)$. Thus a^* is well-defined. And for $\alpha, \beta \in u^{\top}(A, C)$, $\alpha^*(\alpha \dot{+} \beta) = \{\mu \circ (\alpha \times \beta) \circ \Delta_A\} \circ a = \mu \circ (\alpha \circ a \times \beta \circ a) \circ \Delta_V = \alpha^*(\alpha) \dot{+} a^*(\beta)$. This shows that a^* is a homomorphism. The remains can be easily checked.

(2) Let $\mu_1\colon C \times C \to C$ and $\mu_2\colon D \times D \to D$ be pairings. For $\alpha, \beta \in u^{\top}(A, C)$,

$$\begin{aligned} d_*(\alpha \dot{+} \beta) &= d \circ (\alpha \dot{+} \beta) \\ &= d \circ \{\mu_1 \circ (\alpha \times \beta) \circ \Delta_A\}. \end{aligned}$$

Since $d\colon C \to D$ is a Hopf map,

$$d \circ \mu_1 \simeq \mu_2 \circ (d \times d).$$

Thus

$$\begin{aligned} d \circ \{\mu_1 \circ (\alpha \times \beta) \circ \Delta_A\} &= \mu_2 \circ (d \times d) \circ (\alpha \times \beta) \circ \Delta_A \\ &= \mu_2 \circ (d \circ \alpha \times d \circ \beta) \circ \Delta_A \\ &= d_*(\alpha) + d_*(\beta). \end{aligned}$$

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