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AN ESTIMATE OF FOCAL POINTS FOR SPACELIKE HYPERSURFACES

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1. Introduction

Recently, P. Ehrlich and S.-B. Kim[5] obtained a condition of the existence of conjugate points for Riemannian or timelike geodesics by using the stable Jacobi tensors and its associated Riccati equation in [8], which is called the Raychaudhuri equation in general relativity. From the Riccati equation, we have a Jacobi equation problem for conjugate points in the oscillation theory of second order linear differential equations (cf. [15,17]).

D. N. Kupeli[13,14] investigated the Riccati equations to obtain conjugate points and focal points along geodesics on semi-Riemannian manifolds. More in details, in [13] he generalized the Myers theorem for the conjugate points by using the Fourier coefficients of the Ricci curvature which is sharper than the Myers-Galloway theorem given in [9], and there he used a differential inequality originated in the index form technique (cf. [1,2,3,4,5,6,7,10,11]) but did not used the Riccati equation involving the shear tensor. Moreover, in [14] he tried to generalize the Myers theorem for the focal point case by using the Riccati equation containing the shear tensor but he restricted the initial condition of Jacobi tensor to get a focal point by setting the second fundamental tensor zero.

On the other hand, for the existence of focal points along timelike geodesics orthogonal to the spacelike submanifold of codimension arbitrary, S.-B. Kim and D.-S. Kim[12] generalized the Myers-Galloway theorem on Lorentzian manifold by using the submanifold index form.

In this paper, using the Riccati equation technique we first extend the restricted Myers theorem given by D. N. Kupeli[14] by including the nonzero second fundamental tensor into the initial condition of K-Jacobi tensors. Second, we show the Myers-Galloway theorem and its diameter

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theorem for the spacelike hypersurface which sharpens the results given by S.-B Kim and D.-S. Kim[12].

2. Preliminaries

Let (M, g) be an arbitrary space-time. Given $p, q \in M$, $p \leq q$ means that $p = q$ or there is a piecewise smooth future directed nonspacelike curve from p to q, and $p \ll q$ means that there is a piecewise smooth future directed timelike curve from p to q.

Let $\gamma : [0,1] \to M$ be a unit timelike geodesic segment. and let K be a spacelike submanifold of dimension $k \geq 0$. For $q \in M$, we notate $K << q$ if there exists $p \in K$ such that $p << q$ and $K \leq q$ if there exists $p \in K$ with $p \leq q$. Let $I^+(K) = \{q \in M | K << q\}$ be the chronological future of K, $I^-(K) = \{q \in M | q << K\}$ the chronological past of K, $J^+(K) = \{q \in M | K \leq q\}$ the causal future of K, and $J^-(K) = \{q \in M | q \leq K\}$ the causal past of K. Clearly, $I^+(K) = \bigcup_{p \in K} I^+(p)$ where $I^+(p) = \{q \in M | p \ll q\}$.

Now, let $\Omega_{K,q}$ be the path space of all piecewise smooth future directed nonspacelike curves $\gamma : [0, b] \to (M, g)$ with $\gamma(0) \in K$ and $\gamma(b) = q$. The *Lorentzian arc length* $L : \Omega_{K,q} \longrightarrow \mathbf{R}$ for a partition $0 = t_0 < t_1 < \ldots\ldots < t_n = b$ such that $\gamma|_{(t_{i-1}, t_i)}$ is smooth for $i = 1, 2, \ldots\ldots, n$ is given by

$$L(\gamma) = \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \sqrt{-g(\gamma'(t), \gamma'(t))} dt.$$

Now we define the *Lorentzian distance* from K to q by

$$d(K, q) = \begin{cases} 0, & \text{if } q \notin J^+(K) \\ sup\{L(\gamma) | \gamma \in \Omega_{K,q}\}, & \text{if } q \in J^+(K). \end{cases}$$

Clearly, $d(K, q) > 0$ iff $q \in I^+(K)$. $q \in J^+(K) - I^+(K)$ implies that $d(K, q) = 0$, But the converse does not hold, since $d(K, q) = 0$ for $q \notin J^+(K)$.

Given a timelike curve γ from K to q, we have a variation α of $\gamma(t)$ and define the variation vector field V of α along γ by

$V(t) = \frac{\partial}{\partial s} \alpha(t, s)|_{s=0}$, $V(b) = 0$, $V(0) \in T_{\gamma(0)} K$.

Then we may recall some facts. If $\gamma : [0, b] \to (M, g)$ is a unit speed timelike geodesic from K to q, then $L'(0) = g(V(0), \gamma'(0))$. Thus, γ is extremal iff γ is orthogonal at $\gamma(0)$ to K.

Moreover, if $\gamma : [0,b] \to (M,g)$ is a unit timelike geodesic which is orthogonal at $\gamma(0)$ to the spacelike submanifold K and assume that V is a piecewise smooth vector field along γ orthogonal to γ', then we have

$$L''(0) = g(S_{\gamma'(0)}V(0), V(0)) + I(V,V)$$

where $I(V,V) = -\int_0^b [g(V',V') - g(R(V,\gamma')\gamma', V)]dt$ and, $S_{\gamma'(0)}$ is the second fundamental tensor given by $S_{\gamma'}x = -(\nabla_x \gamma'(0))^T$ for $x \in T_pK$ where T means "tangential part".

Let $V^{\perp}(\gamma)$ be a vector space of piecewise smooth vector fields Y with orthogonal to γ' and set $V^{\perp}(\gamma,K) = \{Y \in V^{\perp}(\gamma) | Y(0) \in T_{\gamma(0)}K\}$. Then the *Lorentzian submanifold index form*

$$I_{(b,K)} : V^{\perp}(\gamma,K) \times V^{\perp}(\gamma,K) \to \mathbf{R}$$

for $X, Y \in V^{\perp}(\gamma,K)$, is defined by

$$I_{(b,K)}(X,Y) = g(S_{\gamma'(0)}X(0), Y(0)) + I(X,Y)$$

where I is the index form on $V^{\perp}(\gamma)$.

Now a smooth vector field $J \in V^{\perp}(\gamma,K)$ is called a *K-Jacobi field along γ* if J satisfies

(1) $J'' + R(J,\gamma')\gamma' = 0$.

(2) $J'(0) + S_{\gamma'(0)}J(0) \in (T_{\gamma(0)}K)^{\perp}$.

Hence, $\gamma(t_0), t_0 \in (0,b]$ is said to be a *K-focal point* if there is a nontrivial K-Jacobi field with $J(t_0) = 0$.

Let $V_0^{\perp}(\gamma,K)$ be the subspace of $V^{\perp}(\gamma,K)$ with $Y(b) = 0$. Then, from the maximality of K-Jacobi fields along timelike geodesics, we have the following proposition (cf. [11]).

PROPOSITION 2.1. *If $\gamma : [0,b] \to M$ is a future directed timelike geodesic perpendicular to the spacelike submanifold K with no K-focal points. Then the submanifold index form $I_{(b,K)}$ is negative definite on $V_0^{\perp}(\gamma,K) \times V_0^{\perp}(\gamma,K)$.*

Using the submanifold index form $I_{(b,K)}$ it is well known that a timelike geodesic orthogonal to a spacelike submanifold K fails to maximize arc length after the first K-focal point (cf. [1,2,12]).

Moreover, if (M,g) is a globally hyperbolic space-time, we know that there is a future directed maximal nonspacelike geodesic between any causally related two points. However, we can not guarantee the existence of the future directed maximal nonspacelike geodesic from K to a point in M (even if K is closed)[12].

If M is globally hyperbolic and if $J^-(q) \cap K$ is compact, then the function $x \to d(x,q)$ is continuous on the compact set $J^-(q) \cap K$. Hence, it has a maximum at $p \in J^-(q) \cap K$. Thus, $d(K,q) = d(p,q)$. Therefore, there is a geodesic γ from p to q of length $d(K,q) = d(p,q)$. We may assume that $q \notin K$ and $p << q$. From the first variation formula, it is normal to K. Thus we have the following proposition[12].

PROPOSITION 2.2. *Let (M,g) be a globally hyperbolic space-time and let K be a spacelike submanifold of (M,g). Then for any $q \in I^+(K)$ with $J^-(q) \cap K$ compact, there is a future directed maximal timelike geodesic γ perpendicular at $\gamma(0)$ to K in $\Omega_{K,q}$.*

S.-B. Kim and D.-S. Kim[12] generalized the Myers-Galloway theorem to the K-focal sense by using the sectional curvature and the second fundamental form conditions.

THEOREM 2.3. *Let (M,g) be a space-time of dimension ≥ 2 and γ any unit speed timelike geodesic with length L in $\Omega_{K,q}$ perpendicular at $\gamma(0)$ to the spacelike submanifold K of dimension $k \geq 0$ for any point $q \in M$. Suppose*

$$g(R(u,\gamma'(t))\gamma'(t),u) \geq \frac{1}{n-1}(a + \frac{df}{dt})$$

for all $u \in (\gamma'(t))^\perp$ with $g(u,u) = 1$ along γ, and suppose

$$g(S_{\gamma'(0)}w,w) \geq \frac{f(0)}{n-1}$$

for all $w \in T_{\gamma(0)}K$ with $g(w,w) = 1$, where $a > 0, c \geq 0$ and f is a differentiable function with $|f(t)| \leq c$. Assume

$$L(\gamma) > \frac{\pi}{a}\left((1 - \frac{k}{2(n-1)})c + \sqrt{(1 - \frac{k}{2(n-1)})^2 c^2 + a(n - 1 - \frac{3k}{4})}\right).$$

Then γ can not be maximal.

COROLLARY 2.4. *Let (M, g) be a globally hyperbolic space-time of dimension $n \geq 2$ and K the compact spcelike submanifold of dimension $k \geq 0$. Suppose there exist constants $a > 0$ and $c \geq 0$ such that for any point $q \in M$, and any unit maximal timelike geodesic γ in $\Omega_{K,q}$ with length L perpendicular at $\gamma(0)$ to K,*

$$g(R(u, \gamma'(t))\gamma'(t), u) \geq \frac{1}{n-1}(a + \frac{df}{dt})$$

for all $u \in (\gamma'(t))^{\perp}$ with $g(u, u) = 1$ along γ and

$$g(S_{\gamma'(0)}w, w) \geq \frac{f(0)}{n-1}$$

for all $w \in T_{\gamma(0)}K$ with $g(w, w) = 1$. where f is some function with $|f(t)| \leq c$ along γ. Then

$$diam_K(M, g)$$
$$\leq \frac{\pi}{a}\left((1 - \frac{k}{2(n-1)})c + \sqrt{(1 - \frac{k}{2(n-1)})^2 c^2 + a(n - 1 - \frac{3k}{4})}\right).$$

REMARK. If K is any compact hypersurface of M, we have

$$diam_K(M, g) \leq \frac{\pi}{2a}(c + \sqrt{c^2 + (n-1)a}).$$

3. Jacobi Tensors and its Riccati Equations

P. E. Ehrlich and S.-B. Kim [5], D. N. Kupeli [14] studied on Jacobi tensors and obtained conditions of the existence of conjugate points along Riemannian or timelike geodesics. In this section some definitions and facts of Jacobi tensors follow from [2]. Given any unit speed timelike geodesic $\gamma : [0, L] \longrightarrow (M, g)$, let $(\gamma'(t))^{\perp} = \{v \in T_{\gamma(t)}M | (g(v, \gamma'(t)) = 0\}$ and let $V^{\perp}(\gamma) = \bigcup_{0<t<b}(\gamma(t))^{\perp}$. We recall that *Jacobi tensor field* along γ is a smooth (1,1) tensor field $A : V^{\perp}(\gamma) \rightarrow V^{\perp}(\gamma)$ which satisfies

$$A'' + RA = 0 \tag{3.1}$$

and

$$Ker(A) \cap Ker(A') = 0$$

for all $t \in (a, b)$ where $RA(v) = R(A(v), \gamma')\gamma'$. If A is a Jacobi tensor field along γ and P is a parallel field along γ, then $J = A(P)$ is a Jacobi vector field along γ.

Given a Jacobi tensor field A along γ, set $B = A'A^{-1}$ at which A^{-1} is defined. One defines the *expansion tensor* θ = trB, the *vorticity tensor* $\omega = \frac{1}{2}(B - B^*)$, and the *shear tensor* $\sigma = \frac{1}{2}(B + B^*) - (\theta/n-1)Id$. Here B^* is the adjoint tensor field defined by requiring that $g(B^*v, w) = g(v, B(w))$ for all $v, w \in V(\gamma(t))$. Thus $B = \omega + \sigma + (\theta/(n-1))Id$ and it may also be shown that $\theta = tr(A'A^{-1}) = (\det(A))'/\det(A)$. The Jacobi equation (3.1) is converted into the associated Raychaudhuri equation by calculating θ'. Explicitly, one obtains

$$\theta' + Ric(\gamma', \gamma') + tr(\omega^2) + tr(\sigma^2) + \frac{\theta^2}{n-1} = 0. \tag{3.2}$$

If the Jacobi tensor field A satifies the initial condition $A(0) = 0$, $A'(0) = Id$, then $B = B^*$ so that $\omega = 0$ and equation (3.1) simplifies to

$$-\theta' = Ric(\gamma', \gamma') + tr(\sigma^2) + \frac{\theta^2}{n-1}. \tag{3.3}$$

Now the change of variables $x = (detA)^{\frac{1}{n-1}}$ is made. Then we obtains $x'' + \frac{1}{n-1}(Ric(\gamma', \gamma') + tr(\sigma^2))x = 0$. Therefore, it can be shown that $x = (detA)^{\frac{1}{n-1}}$ satisfies the differential equation $x'' + \dfrac{1}{n-1}(Ric(\gamma', \gamma') + tr\sigma^2)x = 0$ with the initial condition $x(0) = 0$ and $x'(0) = (detA'(0))^{\frac{1}{n-1}}$. Hence, for a given Jacobi tensor A satisfying the above initial condition, $x'' + \dfrac{1}{n-1}(Ric(\gamma', \gamma') + tr\sigma^2)x = 0$ can be considered as a Jacobi differential equation for the conjugate points on the interval $(0, b]$.

In [14], D. N. Kupeli proved a Myers type theorems by using a second order linear differential equation including a term of the shear tensor. Let

$$a_0 = \frac{2}{(n-1)L}\int_0^L [Ric(\gamma', \gamma') + tr\sigma^2]dt$$

and

$$a_1 = \frac{2}{(n-1)L}\int_0^L [Ric(\gamma', \gamma') + tr\sigma^2]\cos(\frac{2\pi t}{L})dt$$

be the Fourier coefficients of $Ric(\gamma', \gamma') + tr\sigma^2$ where $tr\sigma^2$ can be extended to a continuous function on $[0, L]$. But, in [13] he had extended the Myers theorem by using the Fourier coefficients of just $Ric(\gamma', \gamma')$.

THEOREM 3.1. *(Myers) Let* $\gamma : [0, L] \to M$ *be a geodesic segment. If* $L^2(a_0 - a_1) \geq 2\pi^2$ *then* γ *contains a conjugate point to "0". In particular,* L *is the first conjugate point to "0" along* γ *if and only if* $\frac{1}{n-1}(Ric(\gamma', \gamma') + tr(\sigma^2)) = (\frac{\pi}{L})^2$.

Moreover, in [13] the above theorem further gives a generalization of the Myers theorem given by Galloway[9].

THEOREM 3.2. *(Myers-Galloway) Let* $\gamma : [0, L] \to M$ *be a unit timelike geodesic on a Lorentzian manifold* M*. Assume that there exist constants* $a > 0$, $c \geq 0$ *and a differentiable function* $f : [0, L] \to \mathbf{R}$ *with* $|f| \leq c$ *such that* $Ric(\gamma', \gamma') \geq (a + \frac{df}{dt})$*. Then* γ *contains a conjugate point to "0" if* $L \geq (2c + \sqrt{4c^2 + (n-1)a\pi^2})/a$*. In particular,* L *is the first conjugate point to "0" along* γ *if and only if* $R_\gamma = (\pi^2/L^2)Id$*, where* $L = \big(2c + \sqrt{4c^2 + a(n-1)\pi^2}\big)/a$.

REMARK. From the above Theorem 3.2, D. N. Kupeli in fact showed that the inequality $L \geq \big(2c + \sqrt{4c^2 + a(n-1)\pi^2}\big)/a$ is sharper than the inequality $L \geq \big(\pi c + \sqrt{\pi^2 c^2 + a(n-1)\pi^2}\big)/a$ in [9].

4. K-Jacobi Tensors and Main Results

Let M be a Lorentzian manifold of dimension $n \geq 2$ and K be a spacelike hypersurface and let $\gamma : [0, L] \longrightarrow M$ be a unit speed timelike geodesic perpendicular to K at $p = \gamma(0)$. We define the *K-Jacobi tensor* A along γ by a Lagrange Jacobi tensor along γ satisfying

$$(1)\ A'' + RA = 0$$
$$(2)\ A(0) = Id, A'(0) = -S_{\gamma'(0)}.$$

Set $E_n(t) = \gamma'(t)$ and let $E_1, E_2, \cdots, E_{n-1}$ be $n-1$ spacelike parallel orthonormal frame fields along γ such that $E_1(0), E_2(0), \cdots, E_{n-1}(0)$ forms on basis of T_pK.

Setting $B = A'A^{-1}$ whenever A is non-singular, $\theta = trB$, $\sigma = B - \frac{1}{n-1}\theta Id$, and using a matrix Riccati type equation $B' = A''A^{-1} - A'A^{-1}A'A^{-1} = -R - B^2$, we obtain as in section 3 a traced Riccati equation

$$\theta' + \frac{1}{n-1}\theta^2 + Ric(\gamma', \gamma') + tr\sigma^2 = 0.$$

Put $f = Ric(\gamma', \gamma') + tr\sigma^2$, then we obtain a standard Riccati equation $\theta' + a\theta^2 + f = 0$ where $a = \dfrac{1}{n-1}$. Set $y = (\det A)^{\frac{1}{n-1}}$. Then $\dfrac{\theta}{n-1} = \dfrac{y'}{y}$ $(y \neq 0)$. Since $\dfrac{1}{n-1}\theta' = \dfrac{y''y - y'^2}{y^2}$, by substitution into Ricatti equation, we obtain $y'' + \dfrac{1}{n-1} fy = 0$ with $y(0) = 1, y'(0) = -\dfrac{1}{n-1} trS_{\gamma'(0)}$.

Thus, γ contains a K-focal point at $t_0 \in (0, L]$ to M if and only if $y'' + \dfrac{1}{n-1} fy = 0$ has a nontrivial solution y with $y(t_0) = 0$.

Now, we extend a focal Myers theorem for a spacelike hypersurface given in [14] by including the nonzero trace of second fundamental tensor as follows.

THEOREM 4.1. *Let $\gamma : [0, L] \to M$ be a unit timelike geodesic orthogonal to the spacelike hypersurface K. Let*

$$b_0 = \frac{2}{(n-1)L} \int_0^L [Ric(\gamma', \gamma') + tr\sigma^2] dt$$

and let

$$b_1 = \frac{2}{(n-1)L} \int_0^L [Ric(\gamma', \gamma') + tr\sigma^2] \cos(\frac{\pi t}{L}) dt$$

if $tr\sigma^2$ can be extended to a continuous function on [0,L]. Then, γ contains a K-focal point to K if $L^2(b_0 + b_1) \geq \dfrac{\pi^2}{2} - \dfrac{4L}{n-1} trS_{\gamma'(0)}$.

Proof. Let $E_1(t), E_2(t), \cdots, E_{n-1}(t)$ be $n-1$ spacelike parallel fields along γ. Set $W_i = \cos(\frac{\pi t}{2L}) E_i$ for $i = 1, 2, \cdots, n-1$. Since $E_i(0) \in T_{\gamma'(0)}K$ and since $W_i(L) = 0$, $W_i \in V_0^{\perp}(\gamma, K)$. By computing the submanifold index form, we have

$$\begin{aligned} &I_{(L,K)}(W_i, W_i) \\ &= g(S_{\gamma'(0)} W_i(0), W_i(0)) - \int_0^L [(\frac{\pi}{2L})^2 \sin^2(\frac{\pi t}{2L}) g(E_i, E_i) \\ &- \cos^2(\frac{\pi t}{2L})(g(R(E_i, \gamma')\gamma', E_i) + g(\sigma^2 E_i, E_i))] dt \end{aligned}$$

$i = 1, 2, \cdots, n-1$. Since $W_i(0) = E_i(0)$ we sum up the above index form for all $i = 1, 2, \cdots, n-1$. Then, since $\int_0^L \cos(\frac{\pi t}{L})dt = 0$, we obtain

$$\begin{aligned} & trS_{\gamma'(0)} + \int_0^L [\cos^2(\frac{\pi t}{2L})(Ric(\gamma',\gamma') + tr\sigma^2) \\ & -(n-1)(\frac{\pi}{2L})^2 \sin^2(\frac{\pi t}{2L})]dt \\ & = trS_{\gamma'(0)} - (n-1)\frac{\pi^2}{8L} + \frac{1}{2}\int_0^L [Ric(\gamma',\gamma') + tr\sigma^2]dt \\ & + \frac{1}{2}\int_0^L [Ric(\gamma',\gamma') + tr\sigma^2]\cos(\frac{\pi t}{L})dt \\ & = trS_{\gamma'(0)} - (n-1)\frac{\pi^2}{8L} + \frac{L}{4}(n-1)(b_0 + b_1) \end{aligned}$$

If the submanifold index form $I_{(L,K)}$ is semi-positive on $V_0^{\perp}(\gamma, K)$, there is a K-focal point along γ. Thus, we obtain

$$L^2(b_0 + b_1) \geq \frac{\pi^2}{2} - \frac{4L}{n-1} trS_{\gamma'(0)}$$

as a necessary condition to get a K-focal point along γ.

Thus, we may obtain a generalized Myers-Galloway theorem which is sharper than Theorem 3.2 given in [13].

THEOREM 4.2. *Let $\gamma : [0, L] \to M$ be a unit timelike geodesic orthogonal at $\gamma(0)$ to the spacelike hypersurface K. Assume that there exist constant $a > 0, c \geq 0$ and a differentiable function $f : [0, L] \to \mathbf{R}$ with $|f| \leq c$ such that $Ric(\gamma', \gamma') \geq a + \frac{df}{dt}$ and $trS_{\gamma'(0)} \geq f(0)$. Then γ contains a K-focal point if $L \geq \frac{1}{a}\left(c + \sqrt{c^2 - \frac{(n-1)a\pi^2}{4}}\right)$.*

Proof. Since $Ric(\gamma', \gamma') \geq a + \frac{df}{dt}$, $trS_{\gamma'(0)} \geq f(0)$, and $tr\sigma^2 \geq 0$, a straight forward computation shows that

$$\begin{aligned} b_0 + b_1 &= \frac{2}{(n-1)L}\int_0^L [Ric(\gamma',\gamma') + tr\sigma^2](1 + \cos(\frac{\pi t}{L}))dt \\ &\geq \frac{2}{(n-1)L}\int_0^L (a + \frac{df}{dt})(1 + \cos(\frac{\pi t}{L}))dt \\ &= \frac{2}{(n-1)L}[aL - 2f(0) + \frac{\pi}{L}\int_0^L f(t)\sin(\frac{\pi t}{L})dt]. \end{aligned}$$

An estimation shows that

$$|\int_0^L [f(t)\sin(\frac{\pi t}{L})dt| \leq \frac{2cL}{\pi}.$$

Hence,

$$L^2(b_0+b_1) \geq \frac{2}{n-1}(aL^2 - 2trS_{\gamma'(0)}L - 2cL).$$

Thus, from Theorem 4.1, γ contains a K-focal point if

$$\frac{2}{n-1}(aL^2 - 2trS_{\gamma'(0)}L - 2cL) \geq \frac{\pi^2}{2} - \frac{4L}{n-1}trS_{\gamma'(0)}.$$

Thus, γ contains a K-focal point if L satisfies the inequality

$$aL^2 - 2cL - \frac{(n-1)\pi^2}{4} \geq 0.$$

Then, it follows that

$$L \geq \frac{1}{a}\left(c + \sqrt{c^2 - \frac{(n-1)a\pi^2}{4}}\right).$$

Hence, we obtain a generalized diameter theorem for the spacelike hypersurfce whose proof is similar to the proof of Theorem 4.2 in [12].

COROLLARY 4.3. *Let (M,g) be a globally hyperbolic space-time and K the compact hypersurface. Suppose there exist constants $a > 0$ and $c \geq 0$ such that for any point $q \in M$, and any unit maximal timelike geodesic γ in $\Omega_{K,q}$ with length L perpendicular at $\gamma(0)$ to K,*

$$Ric(\gamma',\gamma') \geq a + \frac{df}{ds}$$

along γ and

$$trS_{\gamma'(0)} \geq f(0)$$

where f is some function with $|f(0)| \leq c$ along γ. Then

$$diam_K(M,g) \leq \frac{1}{a}\left(c + \sqrt{c^2 - \frac{(n-1)a\pi^2}{4}}\right).$$

REMARK. This diameter is sharper than that given in Corollary 2.4 when K is the spacelike hypersurface since $\frac{\pi}{2} > 1$, and is also just half of that given in Theorem 3.2.

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