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ON THE FIXED POINT FREE DEFORMATION OF THE NILPOTENT SPACES

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1. Introduction

The concept of the nilpotent space was introduced and developed by A.K. Bousfield, G. Mislin, P. Hilton, J. Roitberg and R.H. Lewis ([4], [9], [12]). Furthermore, B. Eckman and G. Mislin have found the concrete examples of the nilpotent space with the relation to the dimension of the topological space ([12], [14]).

In this paper, we shall study the fixed point free deformation of the nilpotent space by use of the Euler characteristic number and consider the following theorems

(1) Let X be an element of S_{*N} which is finite type with $\pi_1(X) \neq \{e\}$. Then X admits a fixed point free deformation (Theorem 3.1).
(2) Let X be an element of S_{*N} which is finite type with $\pi_1(X) \neq \{e\}$. If $f : X \to Y$ is an acyclic map for some space Y, then Y admits a fixed point free deformation (Theorem 3.6).

2. Preliminaries

Throughout this paper, S_* denote the category of the connected CW-complexes and the base point preserving continuous maps and S_{*N} denote the category of the nilpotent spaces and the base point preserving unless otherwise stated continuous maps as the subcategory of S_*.

For any group π and an abelian group G, we can consider the action of π on G as a function $\omega : \pi \to \mathrm{Aut}(G)$. We adopt the abbreviation $x \cdot a$ for $\omega(x)(a)$; $x \in \pi$, $a \in G$. Define *the lower central ω-series of* G;

$$\Gamma^1_\omega(G) \supseteq \cdots \supseteq \Gamma^i_\omega(G) \supseteq \Gamma^{i+1}_\omega(G) \supseteq \cdots\cdots$$

by setting $\Gamma^1_\omega(G) = G$, $\Gamma^{i+1}_\omega(G)$=the group generated by the set $\{\, x \cdot a - a \mid x \in \pi, a \in \Gamma^i_\omega(G) \,\}$, for $i \geq 1$.

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Notice that if $I\pi$ is the augmentaton ideal of the integral group ring $\mathbf{Z}\pi$, then

$$\Gamma_\omega^{i+1}(G) = (I\pi)^i G \qquad ([9]).$$

In fact, each $\Gamma_\omega^i(G)$ is a submodule of G.

We say that π *acts nilpotently on* G if $\Gamma_\omega^j(G) = \{ e \}$ for some $j \in \mathbf{N}$, where e is the identity element of G. If c is the largest integer for which $\Gamma_\omega^c(G) \neq \{ e \}$, we say that ω has *a nilpotency class* c and write $\mathrm{nil}(\omega) = c$. A group G is called *nilpotent* if it has a lower central series

$$G = G_1 \supset G_2 \supset \cdots \supset G_n = \{ e \}$$

where e is the identity element of G such that G_j/G_{j+1} is contained in the center of G/G_{j+1} for all j ([17]). In fact, a group G is nilpotent if and only if the action of G on itself via innerautomorphism $x \cdot g = xgx^{-1}$ $(\forall x, g \in G)$ is nilpotent.

DEFINITION 2.1. For $X \in S_*$, X is *a nilpotent space* if

(1) $\pi_1(X, x_0)$ is a nilpotent group

(2) $\pi_1(X, x_0)$ acts on $\pi_n(X, x_0)$ nilpotently for $n \geq 2$.

We can make $\pi_1(X)$ a nilpotent group as follows:

LEMMA 2.2. *If U is normal subgroup of G, then $[U, G]$ is normal subgroup of G, where $[\ ,\ ]$ menas a commutator subgroup.*

Proof. For any $v \in U$ and $\sigma, \tau \in G$, we have $\sigma \cdot v\tau v^{-1}\tau^{-1} \cdot \sigma^{-1} = \sigma v\sigma^{-1} \cdot \sigma\tau\sigma^{-1} \cdot (\sigma v\sigma^{-1})^{-1} \cdot (\sigma\tau\sigma^{-1})^{-1} \in [U, G]$. Since $\sigma v\sigma^{-1} \in U$, for any $x = g_1 \cdots g_n \in [U, G]$ where $g_i = v_i\tau_i v_i^{-1}\tau_i^{-1}$, $v_i \in U$, $\tau_i \in G$, we have

$$\sigma x\sigma^{-1} = \sigma g_1\sigma^{-1} \cdot \sigma g_2\sigma^{-1} \cdots \sigma g_n\sigma^{-1} \in [U, G].$$

In order to make $\pi_1(X)$ a nilpotent group, we consider the following lower central series

$$\begin{array}{ccccccccc} \pi_1(X) & \supset & [\pi_1(X), \pi_1(X)] & \supset & [G_1, \pi_1(X)] & \cdots & \supset & G_j & \supset \cdots \\ \| & & \| & & \| & & & & \\ G & & G_1 & & G_2 & & & & \end{array}$$

where $G_j = [G_{j-1}, \pi_1(X)]$ for any $j \geq 1$. If there exists n such that $G_n = \{ e \}$, then $\pi_1(X)$ is a nilpotent group.

Notice that G_j/G_{j+1} lies in the center of G/G_{j+1} ([17]) and by Lemma 2.2. for any G_j is normal subgroup of $\pi_1(X)$.

DEFINITION 2.3. A CW-complex with a single nonvanishing homotopy group π occuring in dimension n is called *an Eilenberg-MacLane space* $k(\pi, n)$.

It is well-known that $X \in S_{*N}$, $\pi_1(X)$ acts on the covering transformation on $\tilde{X}$ and $\pi_1(X)$ acts on the integral homology group $H_*(\tilde{X}; \mathbf{Z})$, where $\tilde{X}$ is the universal convering space of X. Hereafter, we shall denote the universal covering space of X by $\tilde{X}$.

DEFINITION 2.4. A map $f: X \longrightarrow X$ is called *a fixed point free deformation* if f has no fixed point and is homotopic to 1_X.

LEMMA 2.5. *A space X is nilpotent if and only if $\pi_1(X)$ is a nilpotent group and the $\pi_1(X)$-module $H_i(\tilde{X}; \mathbf{Z})$ are nilpotent for all $i \geq 0$, ([9],[14]).*

LEMMA 2.6. *If X is a polyhedron and $\chi(X) = 0$, then X admits a fixed point free deformation ([3]).*

LEMMA 2.7. *If $G = \pi_1(X)$ contains a torsion free normal abelian subgroup $A \neq 1$ which acts nilpotently on $H_*(\tilde{X})$, then $\chi(X) = 0$ ([6]).*

3. Main Theorems

In this section, we shall study the fixed point free deformation of the nilpotent space by use of the Euler characteristic number.

Recall that for a nilpotent space X, the following conditions are equivalent ([8],[13]):

(1) $H_i(X; \mathbf{Z})$ is finitely generated for all $i \geq 0$.
(2) $\pi_i(X)$ is finitely generated for all $i \geq 0$.
(3) X has the homotopy type of a complex with finite skeleton.

If one of those equivalent conditions is satisfied, we say that X is *finite type*.

We would like to find some special spaces that admit a fixed point free deformation. Namely, we consider the special space that satisfies converse of the Lefschetz fixed point free deformation.

THEOREM 3.1. *Let X be an element of S_{*N} which is finite type with $\pi_1(X) \neq \{e\}$. Then X admits a fixed point free deformation.*

Proof. Since $X \in S_{*N}$ and X is finite type, $\pi_1(X)$ is finitely generated nilpotent group and by Lemma 2.5, $\pi_1(X)$ acts nilpotently on $H_i(\tilde{X})$. Here we consider the $\pi_1(X)$ as two cases; namely, either $\pi_1(X)$ is infinite or finite.

Suppose that $\pi_1(X)$ is infinite. Since $\pi_1(X)$ is an infinite finitely generated nilpotent group, the center $Z(\pi_1(X))$ of $\pi_1(X)$ is infinite and finitely generated. Then we can take an infinite cyclic subgroup of $Z(\pi_1(X))$ for a torsion-free normal abelian subgroup $A(\neq 1)$ of Lemma 2.7. Thus $\chi(X) = 0$

Now asssume that $\pi_1(X)$ is finite. Mislin and Lewis prove that if $\pi_1(X)$ is finite, then $\chi(\tilde{X}) = \chi(X)$ ([12],[14]). On the other hand, $\chi(X) = |\pi_1(X)|\chi(\tilde{X})$ ([]), where $|\ |$ means the order of $\pi_1(X)$. Since $|\pi_1(X)| \neq 0$, $\chi(X) = 0$. By Lemma 2.6, the space X above admits a fixed point free deformation.

COROLLARY 3.2. *Let X be an element of S_{*N} which is finite type and $\pi_1(X) \neq \{e\}$ is finite. Then $\tilde{X}$ also admits a fixed point free deformation.*

Proof. If follows from Lemma 2.6 and Theorem 3.1.

We recall the Gottlieb Theorem : If X is a connected aspherical polyhedron and the Euler number $\chi(X)$ is not zero, then the center of $\pi_1(X)$ is the trivial group ([9]).

W. Lopes proves the fact : Let X be finite polyhedron with the property that no finite collection of points seperates X. Then X admits a fixed point free deformation if $\chi(X) = 0$ ([13]).

For a given continuous map f between CW-complexes X and Y, we denote the homotopy fiber F_f as the pullback of the following commutative diagram

$$\begin{array}{ccc} F_f & \longrightarrow & PY \\ \downarrow & & \downarrow{\scriptstyle \pi_Y} \\ X & \xrightarrow{f} & Y \end{array}$$

where PY is the conractible space of paths $\lambda : I \longrightarrow Y$ which start at the base point and whose terminal point is given by $\pi_Y(\lambda)$.

DEFINITION 3.3. A map $f : X \to Y$ is called *an acyclic* if its homotopy fiber F_f is an acyclic space, i. e., $\tilde{H}_*(F_f) = 0$.

DEFINITION 3.4. A map $f : X \to Y$ is called *a weak homotopy equivalence* if $\pi_i(f) : \pi_i(X, x) \to \pi_i(Y, f(x))$ is one-to-one and onto for all $i \geq 0$ and all $x \in X$

LEMMA 3.5. *(Whitehead Theorem) Let X and Y be CW-complexes. Assume that $f : X \to Y$ is a weak homotopy equivalence. Then f is a homotopy equivalence ([7]).*

THEOREM 3.6. *Let $X \in S_{*N}$ be finite type with $\pi_1(X) \neq \{e\}$. If $f : X \to Y$ is an acyclic map, then Y also admits a fixed point free deformation.*

Proof. We must prove that X and Y are the same homotopy type; then by use of Theorem 3.1 our conclusion follows.

By assumption, we can consider the following homotopy exact sequence

$$\longrightarrow \pi_2(Y) \longrightarrow \pi_1(F_f) \longrightarrow \pi_1(X) \longrightarrow \pi_1(Y) \longrightarrow \pi_0(F_f).$$

Since f is an acyclic map, $\tilde{H}_0(F_f) = 0 = \pi_0(F_f)$ and $\pi_1(f)$ is an epimorphism. Furthermore

$$\tilde{H}_1(F_f) = H_1(F_f) \cong \pi_1(F_f)/[\pi_1(F_f), \pi_1(F_f)] = 0,$$

where [,] means the commutator subgroup. Hence $\pi_1(F_f)$ is a perfect group. Since the homomorphic image of perfect group is also a perfect group, $\pi_1(Y)$ is isomorphic to $\pi_1(X)/p\pi_1(X)$, where $p\pi_1(X)$ is a perfect normal subgroup of $\pi_1(X)$. Since X is nilpotent, $p\pi_1(X)$ is trivial ([2]). Thus $\pi_1(f)$ is an isomorphism.

Since $f : X \to Y$ is acyclic and $\pi_1(f)$ is an isomorphism, $\pi_1(F_f) = 0$ By the application of the Hurewicz Theorem ([10]) inductively, $\pi_i(F_f) = 0$. Thus $\pi_i(f)$ is an isomorphism for $i \geq 1$. By Lemma 3.5, f is a homotopy equivalence. Since the Euler characteristic is invariant under homotopy equivalence, $\chi(Y) = \chi(X) = 0$. By Lemma 2.6, we conclude that Y also admits the fixed point free deformation.

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