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FINITE TYPE CURVES IN THE LORENTZ MINKOWSKI PLANE

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1. Introduction

Let L_1^m be the m-dimensional Lorentz Minkowski space with metric tensor g_0 given by

$$g_0 = \sum_{j=1}^{m-1} dx_j^2 - dx_m^2,$$

where $(x_1, \cdots, x_m)$ is a rectangular coordinate system of L_1^m. (L_1^m, g_0) is a flat Lorentz manifold of signature $(m-1, 1)$. Let $c = (c_1, \cdots, c_m)$ be a point in L_1^m and $r > 0$. We put

$$S_1^{m-1}(c,r) = \{x \in L_1^m :< x-c, x-c >= r^2\},$$

$$H^{m-1}(c,r) = \{x \in L_1^m :< x-c, x-c >= -r^2, x_m > c_m\},$$

where $< , >$ denote the indefinite inner product on L_1^m. $S_1^{m-1}(c,r)$ and $H^{m-1}(c,r)$ are called the de Sitter space time and the hyperbolic space, respectively ([2,4,10]).

Let $x : M \to L_1^m$ be an isometric immersion from n-dimensional pseudo-Riemannian submanifold M into L_1^m. Denote by Δ the Laplacian of M associated with the pseudo-Riemannian metric on M. The submanifold M of L_1^m is said to be of k-type if the position vector x of M in L_1^m has the following form ([2,3]):

$$x = c + x_{i_1} + \cdots + x_{i_k}, \tag{1.1}$$

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where

$$\Delta x_{i_t} = l_{i_t} x_{i_t}, \quad l_{i_1} < \cdots < l_{i_k},$$

c is a constant vector and $x_{i_1}, \cdots, x_{i_k}$ are non-constant L_1^m-valued eigenfunctions of Δ. A submanifold M is said to be of finite type if it is of k-type for some k. Otherwise, M is said to be of infinite type.

B.-Y.Chen proved in [1] that every closed Euclidean plane curve of finite type is of 1-type, and hence a circle. Later, in [6] B.-Y.Chen, F.Dillen, L.Verstraelen and L.Vrancken proved that a Euclidean plane curve is of finite type if and only if it is an open part of a circle or a straight line.

In this article we study the Lorentzian version of the above. As a result, we prove that every finite type Lorentzian plane curve is of 1-type.

2. Finite type curves in L_1^2

Let $x : \mathbb{R} \to L_1^2$ be a curve parametrized by arclength s. Then the Laplacian Δ of x is given by $\Delta = -\partial^2/\partial s^2$. If x is of finite type in L_1^2, then x can be written as ([6]):

$$\begin{aligned} x(s) = a_0 + as &+ \sum_{j=1}^{m} \{b_j \cos(l_j s) + c_j \sin(l_j s)\} \\ &+ \sum_{i=1}^{k} \{a_i e^{q_i s} + d_i e^{-q_i s}\}, \end{aligned} \tag{2.1}$$

where $0 < l_1 < \cdots < l_m$ and $0 < q_1 < \cdots < q_k$ are positive real numbers and $a_0, a, b_j, c_j, a_i, d_i$ are vectors in L_1^2 such that for each $j \in \{1, \cdots, m\}$ and $i \in \{1, \cdots, k\}$, a_i and d_i (b_j and c_j, respectively) are not simultaneously zero. Let

$$A(s) = a + \sum_{j=1}^{m} l_j[-b_j \sin(l_j s) + c_j \cos(l_j s)]. \tag{2.2}$$

Then we have

$$x'(s) = A(s) + \sum_{i=1}^{k} q_i[a_i e^{q_i s} - d_i e^{-q_i s}]. \tag{2.3}$$

Since the functions $s^{\alpha}e^{\beta s}cos\gamma s$ and $s^{\alpha}e^{\beta s}sin\gamma s$ are linearly independent([8]), $< x'(x), x'(s) >\equiv \pm 1$ is equivalent to the following :

$$H(l) : < A(s), A(s) > -\sum_{i=1}^{k} q_i^2 < a_i, d_i >= \pm 1,$$

$$I(l) : \sum_{\substack{i=1\\2q_i=l}}^{k} q_i^2 < a_i, a_i > +2 \sum_{\substack{i<j\\q_i+q_j=l}} q_i q_j < a_i, a_j >$$
$$+ 2\sum_{\substack{i=1\\q_i=l}}^{k} q_i < A(s), a_i > -2 \sum_{\substack{i<j\\q_j-q_i=l}} q_i q_j < d_i, a_j >= 0,$$

$$J(l) : \sum_{\substack{i=1\\2q_i=l}}^{k} q_i^2 < d_i, d_i > +2 \sum_{\substack{i<j\\q_i+q_j=l}} q_i q_j < d_i, d_j >$$
$$- 2\sum_{\substack{i=1\\q_i=l}}^{k} q_i < A(s), d_i > -2 \sum_{\substack{i<j\\q_j-q_i=l}} q_i q_j < d_j, a_i >= 0,$$

for all $l \in \{q_i|1 \le i \le k\} \cup \{q_i+q_j|1 \le i \le j \le k\} \cup \{q_j-q_i|1 \le i < j \le k\}$.

(Case 1) $k \ge 1$. For $l = 2q_k$ we obtain from $(I(l))$ and $(J(l))$ that $< a_k, a_k >=< d_k, d_k >= 0$. And, for $l = (q_k + q_{k-1})$, $(I(l))$ and $(J(l))$ imply $< a_k, a_{k-1} >=< d_k, d_{k-1} >= 0$.

There are three possibilities:

(1) $a_k = \alpha_k(1,1) \neq 0$, $d_k = \delta_k(1,1) \neq 0$. We obtain $a_{k-1} = \alpha_{k-1}(1,1)$ and $d_{k-1} = \delta_{k-1}(1,1)$. Now, suppose that $a_t = \alpha_t(1,1)$ and $d_t = \delta_t(1,1)$ for all $t > r$. We show that a_r and d_r are also of this form. Indeed, we have

$$< a_i, a_j >=< d_i, d_j >= 0 \quad \text{for all } i, j > r.$$

For $l = (q_k + q_r)$ we obtain from $(I(l))$ and $(J(l))$ that

$$< a_k, a_r >=< d_k, d_r >= 0.$$

Thus $a_r = \alpha_r(1,1)$ and $d_r = \delta_r(1,1)$. By induction, $a_i = \alpha_i(1,1)$ and $d_i = \delta_i(1,1)$ for all $i \in \{1, \cdots, k\}$. For $l = q_k$ we obtain from $(I(l))$ and

$(J(l))$ that $< A(s), a_k >= 0$. Thus we obtain $A(s) = \varphi(s)(1,1)$. Hence we have $< x'(s), x'(s) >= 0$. This is a contradiction.

(2) $a_k = \alpha_k(1,1) \neq 0$, $d_k = \delta_k(-1,1) \neq 0$.

We obtain $a_{k-1} = \alpha_{k-1}(1,1)$ and $d_{k-1} = \delta_{k-1}(-1,1)$. Now, suppose that $a_t = \alpha_t(1,1)$ and $d_t = \delta_t(-1,1)$ for all $t > r$. We can have

$$< a_i, a_j >=< d_i, d_j >= 0 \quad \text{for all} \quad i, j > r.$$

For $l = (q_k + q_r)$ $(I(l))$ and $(J(l))$ imply $< a_k, a_r >=< d_k, d_r >= 0$. Thus $a_r = \alpha_r(1,1)$ and $d_r = \delta_r(-1,1)$. By induction, $a_i = \alpha_i(1,1)$ and $d_i = \delta_i(-1,1)$ for all $i \in \{1, \cdots, k\}$. For $l = q_k$ we obtain $< A(s), a_k >=< A(s), d_k >= 0$. Thus we have $A(s) = 0$. And we obtain from $(I(q_j - q_i))$ and $(J(q_j - q_i))$ that

$$< d_i, a_j >=< d_j, a_i >= 0 \quad \text{for all} \quad i < j,$$

so that $\alpha_j \delta_i = \alpha_i \delta_j = 0$ for all $i < j$. Since $\alpha_k \neq 0$ and $\delta_k \neq 0$, for $l = (q_k - q_1)$ we obtain that $\alpha_k \delta_1 = \alpha_1 \delta_k = 0$. Thus $\alpha_1 = \delta_1 = 0$. Similarly, $\alpha_2 = \alpha_3 = \cdots = \alpha_{k-1} = 0$ and $\delta_2 = \delta_3 = \cdots = \delta_{k-1} = 0$, so that $a_1 = d_1 = \cdots = a_{k-1} = d_{k-1} = 0$ and $a_k \neq 0$, $d_k \neq 0$. Hence we may assume that $x(s)$ is of the following form:

$$\begin{aligned} x(s) &= a_0 + a e^{qs} + d e^{-qs} \\ &= a_0 + \alpha(1,1) e^{qs} + \delta(-1,1) e^{-qs}. \end{aligned}$$

Therefore $x(s)$ is congruent to one of the following:

$$\begin{aligned} H^1(r) &= \{x \in L_1^2 \mid < x, x >= -r^2, x_2 > 0\}, \\ S_1^1(r) &= \{x \in L_1^2 \mid < x, x >= r^2, x_1 > 0\}, \end{aligned}$$

where $r = \frac{1}{q} > 0$.

(3) $a_k = a_{k-1} = \cdots = a_{k_0+1} = 0$, $a_{k_0} \neq 0$, $d_k = \delta_k v \neq 0$ where $v = (1,1)$ or $v = (-1,1)$. Then $J(q_k + q_{k-1})$ implies

$$< d_k, d_{k-1} >= 0 \text{ and } d_{k-1} = \delta_{k-1} v.$$

Suppose that $d_t = \delta_t v$ for all $t > r$. For $l = (q_k + q_r)$ we obtain from $(J(l))$ that $< d_k, d_r >= 0$. Thus $d_r = \delta_r v$. Hence $d_i = \delta_i v$ for all $i \in \{1, \cdots, k\}$. For $l = 2q_{k_0}$ we obtain from $(I(l))$ that $< a_{k_0}, a_{k_0} >= 0$, so that $a_{k_0} =$

$\alpha_{k_0}u$ where $u = (1,1)$ or $u = (-1,1)$. For $l = (q_{k_0} + q_{k_0-1})$ we obtain from $(I(l))$ that $< a_{k_0}, a_{k_0-1} >= 0$. Hence $a_{k_0-1} = \alpha_{k_0-1}u$. Suppose that $a_t = \alpha_t u$ for all $k_0 \geq t > r$. Thus $J(q_{k_0} + q_r)$ implies $< a_{k_0}, a_r >= 0$. Hence $a_r = \alpha_r u$. By induction, $a_i = \alpha_i u$ for all $i \in \{1, \cdots, k_0\}$.

(i) If $u \neq v$, then $(I(q_{k_0}))$ and $(J(q_k))$ imply

$$< A(s), a_{k_0} >=< A(s), d_k >= 0.$$

Thus $A(s) = 0$. Hence we have $< x'(s), x'(s) >= 0$. This is a contradiction.

(ii) If $u = v$, then as in (1), we obtain $a_i = \alpha_i v$ for all $i \in \{1, \cdots, k_0\}$ and $d_i = \delta_i v$ for all $i \in \{1, \cdots, k\}$. Thus we obtain from $(I(q_{k_0}))$ and $(J(q_k))$ that $< A(s), a_{k_0} >=< A(s), d_k >= 0$. From this we have $A(s) = \varphi(s)v$. Hence we have $< x'(s), x'(s) >= 0$. This is a contradiction.

(Case 2) $k = 0$. In this case, $x(s)$ is of the following form :

$$x(s) = a_0 + as + \sum_{i=1}^{m} \{b_i \cos(l_i s) + c_i \sin(l_i s)\}.$$

Note that the condition $\langle x'(s), x'(s) \rangle \equiv \pm 1$ is equivalent to the following ([6]) :

$$\sum_{i=1}^{m} l_i^2 D_{ii} = 2(\pm 1 - < a, a >), \tag{2.4}$$

$$-4 \sum_{\substack{i=1 \\ l_i = l}}^{m} l_i M_i + \sum_{\substack{i=1 \\ 2l_i = l}}^{m} l_i^2 A_{ii} + 2 \sum_{\substack{i>j \\ l_i + l_j = l}} l_i l_j A_{ij} - 2 \sum_{\substack{i>j \\ l_i - l_j = l}} l_i l_j D_{ij} = 0, \tag{2.5}$$

$$4 \sum_{\substack{i=1 \\ l_i = l}}^{m} l_i \overline{M}_i + \sum_{\substack{i=1 \\ 2l_i = l}}^{m} l_i^2 \overline{A}_{ii} + 2 \sum_{\substack{i>j \\ l_i + l_j = l}} l_i l_j \overline{A}_{ij} + 2 \sum_{\substack{i>j \\ l_i - l_j = l}} l_i l_j \overline{D}_{ij} = 0, \tag{2.6}$$

for all $l \in \{l_i | 1 \leq i \leq m\} \cup \{l_i + l_j | 1 \leq j \leq i \leq m\} \cup \{l_i - l_j | 1 \leq j < i \leq m\}$, where

$$M_i = \langle a, c_i \rangle, \qquad \overline{M}_i = \langle a, b_i \rangle,$$

$$(2.7)\qquad A_{ij} = \langle b_i, b_j\rangle - \langle c_i, c_j\rangle, \qquad D_{ij} = \langle b_i, b_j\rangle + \langle c_i, c_j\rangle,$$

$$\overline{A}_{ij} = \langle b_i, c_j\rangle + \langle b_j, c_i\rangle, \qquad \overline{D}_{ij} = \langle b_i, c_j\rangle - \langle b_j, c_i\rangle,$$

for all $i, j \in \{1, 2, \cdots, m\}$.
Thus we obtain from (2.5) and (2.6) that $< b_m, b_m >=< c_m, c_m >$ and $< b_m, c_m >= 0$. Since $b_m, c_m \in L_1^2$, we have $< b_m, b_m >=< c_m, c_m >= 0$.

There are two possibilities:

(1) $b_m \neq 0$, $c_m \neq 0$.

Since $< b_m, b_m >=< c_m, c_m >= 0, < b_m, c_m >= 0$, we have $b_m = \beta_m v$ and $c_m = \gamma_m v$ for a null vector v in L_1^2. For $l = l_m + l_{m-1}$ (2.5) and (2.6) imply

$$< b_m, b_{m-1} >=< c_m, c_{m-1} > \text{ and } < b_m, b_{m-1} >= - < b_{m-1}, c_m > .$$

And we have

$$< b_m, b_{m-1} >= \beta_m < v, b_{m-1} >,$$

$$< c_m, c_{m-1} >= \gamma_m < v, c_{m-1} > .$$

Let $A =< v, b_{m-1} >$ and $B =< v, c_{m-1} >$. Since $\beta_m A = \gamma_m B$ and $\beta_m B = -\gamma_m A$, $\beta_m^2 A = \beta_m \gamma_m B = -\gamma_m^2 A$. Hence $(\beta_m^2 + \gamma_m^2)A = 0$. Thus $A =< v, b_{m-1} >= 0$. Similarly, $B =< v, c_{m-1} >= 0$. Then $b_{m-1} = \beta_{m-1} v$ and $c_{m-1} = \gamma_{m-1} v$. Now, suppose that $b_t = \beta_t v$ and $c_t = \gamma_t v$ for all $t > r$. For $l = l_m + l_r$ we obtain from (2.5) and (2.6) that $A_{mr} = 0$ and $\bar{A}_{mr} = 0$. Thus $b_r = \beta_r v$ and $c_r = \gamma_r v$. Hence $b_i = \beta_i v$ and $c_i = \gamma_i v$ for all $i \in \{1, \cdots, k\}$. For $l = l_i$ (2.5) and (2.6) imply that $< a, b_i >=< a, c_i >= 0$. Hence $a = \beta_0 v$. Thus we obtain

$$x'(s) = [\beta_0 + \sum_{i=1}^{m} \{l_i(-\beta_i \sin(l_i s) + \gamma_i \cos(l_i s))\}]v.$$

Hence we have $< x'(s), x'(s) >= 0$. This is a contradiction.

(2) $b_m = b_{m-1} = \cdots = b_{m_0+1} = 0$, $b_{m_0} \neq 0$ and $c_m = \gamma_m v \neq 0$ where $v = (1, 1)$ or $v = (-1, 1)$. For $l = l_m + l_{m-1}$ we obtain $< c_m, c_{m-1} >= 0$. Thus $c_{m-1} = \gamma_{m-1} v$. Suppose that $c_t = \gamma_t v$ for all $t > r$. For $l = l_m + l_r$ we obtain from (2.6) that $< c_m, c_r >= 0$. Thus $c_r = \gamma_r v$. Hence $c_i = \gamma_i v$ for all $i \in \{1, \cdots, k\}$. For $l = 2l_{m_0}$ we obtain from (2.5) that $< b_{m_0}, b_{m_0} >= 0$. Thus $b_{m_0} = \beta_{m_0} u$ where $u = (1, 1)$ or $u = (-1, 1)$. For $l = l_{m_0} + l_{m_0-1}$ (2.6) implies that $b_{m_0-1} = \beta_{m_0-1} u$. Suppose that $b_t = \beta_t u$ for all $m_0 \geq t > r$.

For $l = l_{m_0} + l_r$ we obtain $< b_{m_0}, b_r >= 0$. Thus $b_r = \beta_r u$. Hence $b_i = \beta_i u$ for all $i \in \{1, \cdots, m_0\}$.

(i) If $u = v$, then similarly as in (1) of (Case1), we have $b_i = \beta_i u$ for all $i \in \{1, \cdots, m_0\}$ and $c_i = \gamma_i u$ for all $i \in \{1, \cdots, m\}$. For $l = l_m$ and $l = l_{m_0}$ we obtain $< a, c_m >=< a, b_{m_0} >= 0$. Thus $a = \beta_0 u$. Hence we have $< x'(s), x'(s) >= 0$. This is a contradiction.

(ii) If $u \neq v$, then similarly as in (2) of (Case 1), we have $a = 0$. Hence we obtain $< x'(s), x'(s) >= 0$. This is a contradiction.

Thus we know that $m = 0$ and that $x(s)$ is of the following form:

$$x(s) = a_0 + as.$$

Therefore we obtain the following classification theorem :

THEOREM. *Let $x : \mathbb{R} \to L_1^2$ be a curve parametrized by arclength s. Then $x(s)$ is of finite type if and only if $x(s)$ is, up to congruences on L_1^2, an open part of one of the following 1-type curves :*

(1) a straight line

(2) $S_1^1(r)$

(3) $H^1(r)$

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