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ON THE VANISHING PROPERTIES OF THE WALL OBSTRUCTION

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1. Introduction

Since C.T.C. Wall defined the Wall obstruction, V.J. Lal, G. Mislin, R. Oliver([4],[7],[8],[10]) and the others studied the Wall invariant.

In this paper, we shall study the vanishing property of the Wall obstruction in the certain fibrations by use of the Euler characteristic(Theorem 3.3, Theorem 3.4). All spaces are path connected CW-complexes unless otherwise stated.

2. Preliminaries

For a space X, we consider the group ring $\mathbf{Z}\pi_1(X)$. Let $K_0(\mathbf{Z}\pi_1(X))$ denote the Grothendieck group of the group ring $\mathbf{Z}\pi_1(X)$; the quotient of the free abelian group on the set of isomorphism classes of finitely generated projective modules over $\mathbf{Z}\pi_1(X)$ modulo the subgroup generated by the elements of the form $[P\oplus Q]-[P]-[Q]$, where for a finitely generated projective module $M, [M]$ denotes the isomorphism class of M.

DEFINITION 2.1. A space is called of type FP, if the singular chain complex $C_i\tilde{X}$ of the universal covering $\tilde{X}$ of X is chain homotopy equivalent (as $\mathbf{Z}\pi_1(X)$-complex) to a finite projective complex; i.e., a complex $\bar{C}_i$ with $\bar{C}_i = 0$ for i big enough, and with each $\bar{C}_i$ a finitely generated projective $\mathbf{Z}\pi_1(X)$ module.

If X is of type FP, the Wall obstruction $w(X)$ is defined by

$$w(X) = \sum(-1)^i[\bar{C}_i] \in K_0(\mathbf{Z}\pi_1(X))$$

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where $\bar{C}_i$ is a finite projective complex equivalent to $C_i\tilde{X}$, and $[\bar{C}_i]$ denote the class of $\bar{C}_i$ in the projective class group $K_0(\mathbf{Z}\pi_1(X))$.

It is evident that $w(X)$ is independent of the choice of $\bar{C}_i$. Furthermore a space X of type FP is dominated by a finite complex if and only if $\pi_1(X)$ is finitely presented.[5,8]

DEFINITION 2.2. A π-module M is called nilpotent if $I^k M = 0$ for some $k > 0$ where I denote the augmentation ideal of $\mathbf{Z}\pi$. Furthermore a space X is called nilpotent if $\pi_1(X)$ is nilpotent and the $\pi_1(X)$-module $\pi_i(X)$ $(i > 1)$ are all nilpotent.

We know the fact that; if $\pi_1(X)$ is nilpotent then X is type FP if and only if X is finitely dominated.[3]

LEMMA 2.3. *[7,Theorem 2.1] Let $F \xrightarrow{j} E \to B$ be a fibration with F a finitely dominated complex and B a finite complex. Then E is a finitely dominated complex and*

$$w(E) = j_* w(F)\chi(B)$$

where $j_* : K_0(\mathbf{Z}\pi_1(F)) \to K_0(\mathbf{Z}\pi_1(E))$ is the group homomorphism and χ means Euler characteristic.

LEMMA 2.4. *[4,Theorem 3] Let $F \to E \xrightarrow{p} B$ be a fibration and B and F are dominated by finite CW-complex then*

$$p_* w(E) = w(B)\chi(F)$$

where $p_* : K_0(\mathbf{Z}\pi_1(E)) \to K_0(\mathbf{Z}\pi_1(B))$ is the group homomorphsim.

3. Main Results

If a space X is nilpotent, then $\pi_1(X)$ is also nilpotent and for all $i \geq 0$ the $\pi_1(X)$-modules $H_i(\tilde{X}, \mathbf{Z})$ are nilpotent.[6,9]

Next, suppose that $\pi_1(X)$ is nilpotent and operates nilpotently on $H_i(\tilde{X})$ for all i, then we have the followings[7];

(1) A Cartan-Whitehead decomposition of X ;

$$\cdots \to X(m) \to X(m-1) \to \cdots \to X(2) = \tilde{X} \to X.$$

(2) The fibrations

$$K(\pi_m X, m-1) \to X(m+1) \to X(m),$$

where $X(m)$ is $(m-1)$ connected.

(3) $\pi_m X \cong H_m(X(m))$ for $m \geq 2$.

Assume inductively that $\pi_1(X)$ operates nilpotently on $H_i(X(m))$ for all i and all m with $2 \leq m \leq M$. Then $\pi_1(X)$ operates nilpotently on $H_j(K(\pi_M(X), M-1))$. The Serre spectral sequence associated to the fibration

$$K(\pi_M(X), M-1) \to X(M+1) \to X(M)$$

has an E^2-term

$$E^2_{ij} = H_i(X(M);\, H_j(K(\pi_M(X), M-1))$$

which is a nilpotent $\pi_1(X)$-module for every pair (i, j). Hence $\pi_1(X)$ operates nilpotently on $H_k(X(M+1))$ for all k and $\pi_1(X)$ operates nilpotently on $\pi_{M+1}(X) \cong H_{M+1}(X(M+1))$.

With the facts above the lemma is followed.

LEMMA 3.1. *[7,Proposition 2.1] A space X is nilpotent if and only if $\pi_1(X)$ is nilpotent and for all $i \geq 0$ the $\pi_1(X)$-modules $H_i(\tilde{X} : \mathbf{Z})$ are nilpotent.*

LEMMA 3.2. *[2, Theorem] If $\pi_1(X)$ contains a torsion free nontrivial normal abelian subgroup which acts nilpotently on $H_*(\tilde{X})$ then Euler characteristic $\chi(X) = 0$, where X is a finite complex.*

Recall that for a nilpotent space X if $\pi_i(X)$ is finitely generated for $i \geq 0$, we say that X is of finite type.

THEOREM 3.3. *Let $F \to E \xrightarrow{p} B$ be a fibration with B a finitely dominated space. If F is a finite complex, $\pi_1(F) \neq 0$ with F finite type and E is a nilpotent space, then $w(E) \in Ker\, p_*$, where $p_* : K_0(\mathbf{Z}\pi_1(E)) \to K_0(\mathbf{Z}\pi_1(B))$.*

Proof. Since E is a nilpotent space, the fiber F is also a nilpotent space.[1] From the fact that F is finite type, $\pi_1(F)$ is finitely generated. By lemma 3.1 $\pi_1(F)$ acts nilpotently on $H_i(\tilde{F})$. Now we consider the Euler characteristic of F. If $\pi_1(F)$ is infinite, the center $Z\pi_1(F)$ of $\pi_1(F)$ is infinite and finitely generated. Moreover $Z\pi_1(F)$ acts nilpotently on $H_i(\tilde{F})$. By lemma 3.2 $\chi(F) = 0$. Next, if $\pi_1(F)$ is finite we know that $\chi(F) = \chi(\tilde{F})$ and $\chi(\tilde{F}) = |\pi_1(F)|\chi(F)$ where $|\quad|$ means the order of $\pi_1(F)$. Since $\pi_1(F) \neq 0$, $\chi(F) = 0$. In two cases of $\pi_1(F)$ above, $\chi(F) = 0$. By lemma 2.4 $w(E) \in Kerp_*$. □

THEOREM 3.4. *Let $F \to E \to B$ be a fibration with F finitely dominated CW-complexes. If B is a finite nilpotent space with finite type and $\pi_1(B) \neq 0$ then $w(E) = 0$.*

Proof. Under the conditions of B, $\chi(B) = 0$ by the similar methods of Theorem 3.3. Thus $w(E) = 0$ by lemma 2.3. □

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