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ENRIQUES NETS OF QUADRICS

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0. Notations

The following are the notations we are going to use throughout this paper :
Let S be a surface and D a divisor on S.
$\mathcal{O}_S$: the structure sheaf of S.
$\mathcal{O}_S(D)$: the invertible sheaf associated to D.
K_S : the canonical divisor of S.
$H^i(S, \mathcal{O}_S(D))$: the i-th cohomology group of the sheaf $\mathcal{O}_S(D)$.
$q(S) = \dim_{\mathbb{C}} H^1(S, \mathcal{O}_S)$: the irregularity of S.
$p_g(S) = \dim_{\mathbb{C}} H^2(S, \mathcal{O}_S) = \dim_{\mathbb{C}} H^0(S, \mathcal{O}_S(K_S))$: the geometric genus of S.
$\varkappa(S)$: the Kodaira dimension of S.

We work over the complex number field $\mathbf{C}$. Enriques surfaces are smooth algebraic surfaces S with $\varkappa(S) = 0$, $p_g(S) = 0$ and $q(S) = 0$ ([**1**], [**3**], [**5**], [**6**]).

1. Introduction

In our paper [**9**], we investigated a family of normal quintic surfaces F_5 in $\mathbf{P}^3$ which are birationally isomorphic to Enriques surfaces. Let us denote the space of these normal quintic surfaces by $\mathcal{F}$. We characterized the space of Enriques surfaces S obtained from normal quintic surfaces F_5 in $\mathcal{F}$ by a special type of divisors D on S (Theorem 2.5, [**9**]). The special type of divisors D on S which consists of four isolated elliptic curves, not only has an intersection property but also a geometric property (see Figure 1).

We would like to construct such a divisor D on an Enriques surface S using F. Cossec and A. Verra's result ([**1**], [**4**]) which states that the

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étale double covering of *every* Enriques surface, which is a K3 surface, is birationally isomorphic to the intersection of three quadric hypersurfaces in $\mathbf{P}^5$.

The next goal of this paper is to characterize nets of quadrics in $\mathbf{P}^5$ from which we could get Enriques surfaces S with the special type of divisors D which have the configuration in Figure 1, and to count the moduli of these special nets of quadrics. In this way, we get another way of computing the dimension of the space of Enriques surfaces which are birationally isomorphic to normal quintic surfaces F_5 in $\mathcal{F}$ (cf. Corollary 3.5, [**9**]). Finally, we construct one special net of quadrics in $\mathbf{P}^5$ which gives rise to an Enriques surface S with a divisor D with the configuration in Figure 1.

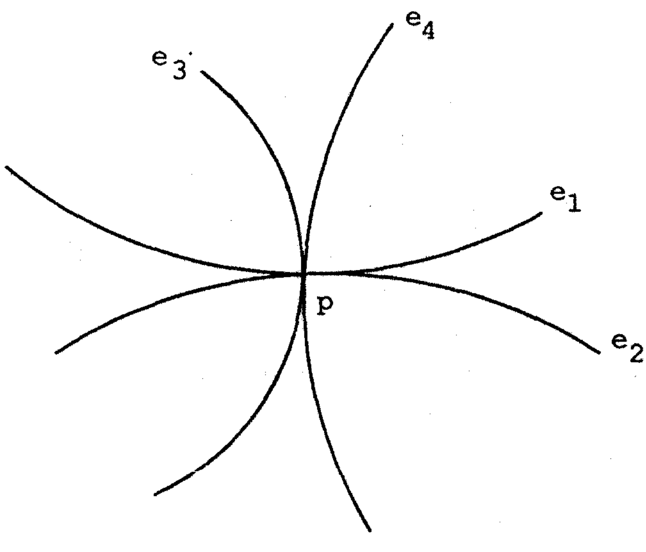

Figure 1

Let S be an Enriques surface, then it is well known ([**1**], [**3**], [**5**], [**6**]) that there exists a unique, non trivial, étale double covering of S, $\pi : T \longrightarrow S$ which is given by the canonical divisor K_S, the only non zero element of order 2 of $Pic(S)$. T is a K3 surface, and it carries a fixed points free involution $\iota_T : T \longrightarrow T$ which interchanges the sheets of the étale double covering π. Moreover, every K3 surface T with a fixed points free involution ι_T is an étale double covering of an Enriques surface S which is the quotient of T by ι_T. This allows Enriques surfaces to correspond bijectively with K3 surfaces endowed with a fixed points free involution.

The following is one of standard examples of constructing Enriques surfaces, which will be used throughout this paper. Let $\mathbf{P}^5$ be a five dimensional projective space over $\mathbb{C}$ with coordinates $x_1, x_2, x_3, y_1, y_2, y_3$. Define an involution ι of $\mathbf{P}^5$ as follows :

$$(1) \qquad \iota(x_1, x_2, x_3, y_1, y_2, y_3) = (-x_1, -x_2, -x_3, y_1, y_2, y_3).$$

Then the set of fixed points of ι is the union of two planes

$$\mathbf{\Pi}_1 = \{y_1 = y_2 = y_3 = 0\} = \{(x_1, x_2, x_3, 0, 0, 0) \in \mathbf{P}^5\},$$
$$\mathbf{\Pi}_2 = \{x_1 = x_2 = x_3 = 0\} = \{(0, 0, 0, y_1, y_2, y_3) \in \mathbf{P}^5\}.$$

We consider three quadrics $\mathbf{Q}_1, \mathbf{Q}_2, \mathbf{Q}_3$ with the following equations :

$$(2) \qquad \mathbf{A}_i(x_1, x_2, x_3) + \mathbf{B}_i(y_1, y_2, y_3) = 0 \;\; (i = 1, 2, 3),$$

where $\mathbf{A}_i, \mathbf{B}_i$ are quadric forms in respective coordinates. These quadrics may be considered as the joins of two plane conics $\mathbf{A}_i \subset \mathbf{\Pi}_1$ and $\mathbf{B}_i \subset \mathbf{\Pi}_2$ fixed by the involution ι, hence $\iota(\mathbf{Q}_i) = \mathbf{Q}_i$ for i=1,2,3. Notice that for a quadric $\mathbf{Q}$ in $\mathbf{P}^5$ which is invariant by the above involution ι, its defining equation is of the form given by (2). Three quadrics $\mathbf{Q}_1, \mathbf{Q}_2, \mathbf{Q}_3$ determine a net of quadrics in $\mathbf{P}^5$, say N, and every quadric in N is also invariant by the involution ι.

Now set $\mathbf{H} = \det(\lambda\mathbf{Q}_1 + \mu\mathbf{Q}_2 + \nu\mathbf{Q}_3) = C_1 \bigcup C_2 \subset \mathbf{P}^2$, the Hessian curve of N, where C_1 and C_2 are plane cubic curves. For a generic net N, its Hessian curve $\mathbf{H}$ has 9 ordinary double points (i.e. double points with distinct tangent directions), which are the intersection of two plane cubic curves C_1 and C_2. These ordinary double points correspond to rank 4 quadrics in $\mathbf{P}^5$. If the Hessian curve $\mathbf{H}$ has only ordinary double points which correspond to rank 4 quadrics, it is then known that the base space of the net N is smooth ([**2**], [**12**]). Thus for a generically chosen net N, the base space of the net N, $T = \mathbf{Q}_1 \cap \mathbf{Q}_2 \cap \mathbf{Q}_3$ is a smooth $K3$ surface. It is easy to see that $T \cap (\mathbf{\Pi}_1 \cup \mathbf{\Pi}_2) = \emptyset$ because $T \cap \mathbf{\Pi}_1, T \cap \mathbf{\Pi}_2$ are the intersection of three plane conics. This says that the induced involution $\iota_T = \iota|_T$ is fixed-point-free. And $\iota(T) = T$ since $\iota(\mathbf{Q}_i) = \mathbf{Q}_i$ for i=1,2,3. Hence we get an Enriques surface S from the K3 surface T.

Through out this paper, we deal only those quadrics which are invariant by the involution ι unless otherwise stated. Thus their quadric forms are given by the equation (2) and their defining matrices are given by

$$(3) \qquad \mathbf{Q} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 & 0 \\ a_{12} & a_{22} & a_{23} & 0 & 0 & 0 \\ a_{13} & a_{23} & a_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & b_{11} & b_{12} & b_{13} \\ 0 & 0 & 0 & b_{12} & b_{22} & b_{23} \\ 0 & 0 & 0 & b_{13} & b_{23} & b_{33} \end{pmatrix}$$

An *Enriques net of quadrics* is a net of quadrics in $\mathbf{P}^5$, where its elements are quadrics invariant by the involution ι and its base locus is a K3 surface with a fixed-point-free involution.

2. Nets of Quadrics

Let $\iota : \mathbf{P}^5 \to \mathbf{P}^5$ be an involution of $\mathbf{P}^5$ given by the equation (1). Then a set $V \subset \mathbf{P}^5$ is called invariant if $\iota(V) = V$, and fixed if $\iota(v) = v$ for all $v \in V$.

Choose an Enriques net of quadrics N so that its Hessian curve $\mathbf{H}$ is the union of two plane cubic curves C_1, C_2 intersecting transversely, and thus has only ordinary double points corresponding to rank 4 quadrics. Let us take one double point $q_1 \in \mathbf{H}$ which is an intersection point of C_1 and C_2. Let $\mathbf{Q}_1$ be a rank 4 quadric in $\mathbf{P}^5$ corresponding to the point q_1. Then $\mathbf{Q}_1$ has the defining matrix of type (2), where its two 3 by 3 submatrices are of rank 2.

Let $\mathbf{A} = \mathbf{Q}_1 \cap \mathbf{\Pi}_1$ and $\mathbf{B} = \mathbf{Q}_1 \cap \mathbf{\Pi}_2$. Then $\mathbf{A}$ and $\mathbf{B}$ are plane conics in $\mathbf{Q}_1$ fixed by the involution ι. Since matrices $\mathbf{A}$ and $\mathbf{B}$ are of rank 2, plane conics $\mathbf{A}$ and $\mathbf{B}$ are the union of two lines of $\mathbf{\Pi}_1$ and $\mathbf{\Pi}_2$ respectively. So set $\mathbf{A} = m_1 \cup m_2$ and $\mathbf{B} = n_1 \cup n_2$, the union of lines. Let $p_1 = m_1 \cap m_2 \in \mathbf{A}$ and $p_2 = n_1 \cap n_2 \in \mathbf{B}$ be the intersection of lines of $\mathbf{A}$ and $\mathbf{B}$. Then the line ℓ generated by two points p_1 and p_2 is the set of singular points of the rank 4 quadric $\mathbf{Q}_1$, that is, $\mathrm{Sing}(\mathbf{Q}_1) = \ell$.

We may consider that the projective five space $\mathbf{P}^5$ is generated by the line ℓ and its complement projective three space, say $\mathbf{P}^3_*$, which is also an invariant space. Let $Q = \mathbf{Q}_1 \cap \mathbf{P}^3_*$. Then Q is a smooth invariant quadric surface. And $\mathbf{Q}_1$ is the join of the singular line ℓ and Q. There are four points a_1, a_2, b_1, b_2 of Q so that

$$m_1 = \overline{p_1 a_1}\,,\ m_2 = \overline{p_1 a_2}\,,\ n_1 = \overline{p_2 b_1}\,,\ n_2 = \overline{p_2 b_2}$$

Four points a_1, a_2, b_1, b_2 are fixed by the involution ι because they belong to four lines m_1, m_2, n_1, n_2 which are also fixed by the involution ι. The smooth quadric surface Q does not have any more fixed points of the involution ι except the above four points.

There is a pair of lines lying in Q at each of points a_1, a_2, b_1, b_2 which are the intersection of Q and tangent planes at $a_1, \ldots, b_2$. Then each of these lines, which is not fixed by the involution ι, contains one fixed point among the four points a_1, a_2, b_1, b_2. Since every one dimensional invariant space contains two fixed points, four pairs of lines at fixed points a_1, a_2, b_1, b_2 are the joins of two points, one from a_1, a_2 and another one from b_1, b_2. Therefore we get all together four invariant lines on Q : $\ell_{a_1b_1}, \ell_{a_1b_2}, \ell_{a_2b_1}, \ell_{a_2b_2}$. It is clear that $\ell_{a_1b_1}, \ell_{a_1b_2}, \ell_{a_2b_1}, \ell_{a_2b_2}$ are the only invariant lines in Q.

The smooth quadric Q has two pencils of lines, γ_1 and γ_2. Then the rank 4 quadric $\mathbf{Q}_1$ has two corresponding pencils of projective three spaces, say Γ_1 and Γ_2, generated by joining the singular line ℓ and lines from the two pencils of lines γ_1 and γ_2.

Now pick a point q of Q. At q there are two lines ℓ_1 and ℓ_2 of Q, that is, $\ell_1 \in \gamma_1, \ell_2 \in \gamma_2$ and $q = \ell_1 \cap \ell_2$. Then we have two projective three space $\mathbf{P}_1^3$ and $\mathbf{P}_2^3$ of $\mathbf{Q}_1$, where $\mathbf{P}_1^3 = \langle \ell, \ell_1 \rangle, \mathbf{P}_2^3 = \langle \ell, \ell_2 \rangle$. Here the notation $\langle A, B \rangle$ means the space generated by A and B.

Let $\mathbf{Q}_1, \mathbf{Q}_2, \mathbf{Q}_3$ be the generators of the net N, where $\mathbf{Q}_1$ is the above rank 4 quadric and $T = \mathbf{Q}_1 \cap \mathbf{Q}_2 \cap \mathbf{Q}_3$. Since $\mathbf{P}_1^3$ and $\mathbf{P}_2^3$ are in $\mathbf{Q}_1$, we have

$$
\begin{aligned}
E_1 &= \mathbf{P}_1^3 \cap (\mathbf{Q}_2 \cap \mathbf{Q}_3) = \mathbf{P}_1^3 \cap (\mathbf{Q}_1 \cap \mathbf{Q}_2 \cap \mathbf{Q}_3) = \mathbf{P}_1^3 \cap T' \\
E_2 &= \mathbf{P}_2^3 \cap (\mathbf{Q}_2 \cap \mathbf{Q}_3) = \mathbf{P}_2^3 \cap (\mathbf{Q}_1 \cap \mathbf{Q}_2 \cap \mathbf{Q}_3) = \mathbf{P}_2^3 \cap T .
\end{aligned}
$$

Hence E_1 and E_2 are quartic space curves with the arithmetic genus 1, so we get two pencils of elliptic curves $|E_1|$ and $|E_2|$ on K3 surface T.

To compute the intersection number of E_1 and E_2, we observe that

$$
E_1 \cap E_2 = (\mathbf{P}_1^3 \cap (\mathbf{Q}_2 \cap \mathbf{Q}_3)) \cap (\mathbf{P}_2^3 \cap (\mathbf{Q}_2 \cap \mathbf{Q}_3)) = \mathbf{P}_{\ell,q}^2 \cap (\mathbf{Q}_2 \cap \mathbf{Q}_3),
$$

where $\mathbf{P}_{\ell,q}^2 = \mathbf{P}_1^3 \cap \mathbf{P}_2^3 = \langle \ell, q \rangle$ a plane generated by the singular line ℓ and the point $q \in Q$. Since $E_1 \cap E_2$ is the intersection of two plane conics, we get the intersection number, $E_1 \cdot E_2 = 4$.

We now claim that there is an action by the induced involution ι on two pencils of elliptic curves $|E_1|$ and $|E_2|$ so that $\iota(|E_1|) = |E_1|$ and $\iota(|E_2|) = |E_2|$. For this purpose, we first check the same claim for two pencils of lines γ_1 and γ_2. Take one pencil of lines γ_1. Then it is easy to see that for a non-invariant line ℓ^*, $\ell^* \in \gamma_1$ implies $\ell^* \cap \iota(\ell^*) = \emptyset$. Thus $\iota(\gamma_1) = \gamma_1$

since $\ell' \cdot \ell'' = 0$ for all $\ell', \ell'' \in \gamma_1$. Similarly for γ_2, and Γ_1, Γ_2. Hence the involution ι induces an order 2 action ι on the pencil of elliptic curves $|E_1|, |E_2|$.

We now assume that the point q is one of four fixed points of Q, for instance a_1 and $\ell_1 \in \gamma_1$ and $\ell_2 \in \gamma_2$ two invariant lines of Q at q, whom we may consider as fixed points of γ_1 and γ_2 by the induced involution ι which acts on γ_1, γ_2. Notice that the singular lines are invariant, hence two projective three spaces generated by the singular line ℓ of $\mathbf{Q}_1$ and two invariant lines ℓ_1 and ℓ_2 are also invariant. Then corresponding elliptic curves E_1, E_2 are invariant too. And we have two pencils of elliptic curves of the Enriques surface $S = T/\iota_T$, $|2e_1|$ and $|2e_2|$, which are induced by $|E_1|, |E_2|$, where, $e_1 = \pi(E_1)$ and $e_2 = \pi(E_2)$. Notice that e_1, e_2 are isolated elliptic curves of S because E_1, E_2 are invariant by the involution ι_T. We now compute the intersection number of e_1 and e_2 as follows : $4 = E_1 \cdot E_2 = 2\ e_1 \cdot e_2 \implies e_1 \cdot e_2 = 2$ We recall that the smooth quadric surface $Q \subset \mathbf{Q}_1$ has exactly four invariant lines $\ell_{a_1b_1}, \ell_{a_1b_2}, \ell_{a_2b_1}, \ell_{a_2b_2}$, where a_1, a_2, b_1, b_2 are all fixed points of Q. Notice that $\ell_{a_1b_1} \cdot \ell_{a_2b_2} = 0$ & $\ell_{a_1b_2} \cdot \ell_{a_2b_1} = 0$, so each of $\ell_{a_1b_1}$ and $\ell_{a_1b_2}$, $\ell_{a_2b_1}$ and $\ell_{a_2b_2}$ belongs to the same pencil of lines on Q, but not all of them are in one pencil of lines. Then choosing different fixed points of Q, we get all together four isolated elliptic curves e_1, e_2, e_1', e_2'.

Now let three quadrics $\mathbf{Q}_1, \mathbf{Q}_2, \mathbf{Q}_3$ in $\mathbf{P}^5$ be generators of the net N, where $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are rank 4 quadrics, which correspond to the intersection points of two cubic curves of the Hessian curve $\mathbf{H}$ of the net N. Then by applying the same process to get two elliptic curves e_1 and e_2 with $e_1 \cdot e_2 = 2$ from the rank 4 quadric $\mathbf{Q}_1$, now we get four elliptic curves e_1, e_2, e_3 and e_4 of S with the intersection number, $e_1 \cdot e_2 = 2$ and $e_3 \cdot e_4 = 2$. Let E_1, E_2, E_3, E_4 be corresponding invariant elliptic curves on K3 surface $T = \mathbf{Q}_1 \cap \mathbf{Q}_2 \cap \mathbf{Q}_3$ and

$$E_1 = \mathbf{P}_1^3 \cap (\mathbf{Q}_2 \cap \mathbf{Q}_3), E_2 = \mathbf{P}_2^3 \cap (\mathbf{Q}_2 \cap \mathbf{Q}_3)$$
$$E_3 = \mathbf{P}_3^3 \cap (\mathbf{Q}_1 \cap \mathbf{Q}_3), E_4 = \mathbf{P}_4^3 \cap (\mathbf{Q}_1 \cap \mathbf{Q}_3),$$

where $\mathbf{P}_1^3, \mathbf{P}_2^3 \subset \mathbf{Q}_1$ and $\mathbf{P}_3^3, \mathbf{P}_4^3 \subset \mathbf{Q}_2$ are invariant three spaces in $\mathbf{P}^5$. We check the intersection number of, for instance, E_1 and E_3 :

$$\begin{aligned} E_1 \cap E_3 &= \mathbf{P}_1^3 \cap (\mathbf{Q}_2 \cap \mathbf{Q}_3) \cap \mathbf{P}_3^3 \cap (\mathbf{Q}_1 \cap \mathbf{Q}_3) \\ &= \mathbf{P}_1^3 \cap \mathbf{P}_3^3 \cap \mathbf{Q}_3 \qquad \text{because } \mathbf{P}_1^3 \subset \mathbf{Q}_1, \mathbf{P}_3^3 \subset \mathbf{Q}_2 \end{aligned}$$

$$= q_1 \cup q_2, \qquad \text{where } q_1, q_2 \text{ are two points of } T$$
$$\Longrightarrow E_1 \cdot E_3 = 2$$

Then $2 = E_1 \cdot E_3 = 2\, e_1 \cdot e_3 \Longrightarrow e_1 \cdot e_3 = 1$. Similarly, $e_1 \cdot e_4 = e_2 \cdot e_3 = e_2 \cdot e_4 = 1$.

Let $D = e_1 + e_2 + e_3 + e_4$. Then D satisfies all conditions of Theorem 2.5 of **[9]** except the geometric property $\mathcal{GP}$. Summerizing what we have observed, we get the following proposition:

PROPOSITION 2.1.. *Every Enriques surface S has a divisor $D = e_1 + e_2 + e_3 + e_4$ with the intersection numbers: $e_1 \cdot e_2 = e_3 \cdot e_4 = 2$, $e_1 \cdot e_3 = e_1 \cdot e_4 = e_2 \cdot e_3 = e_2 \cdot e_4 = 1$, where e_i are elliptic curves ($i = 1, \ldots, 4$).*

3. Characterization

Now we seek to find the condition under which the divisor D has the configuration in Figure 1. First we look for the condition for two elliptic curves e_1, e_2 of S to meet tangentially at a point $p_1 \in S$ and $e_1 \cdot e_2 = 2$. To find this condition, we go back to the construction of e_1, e_2 from a net of quadrics $N = \langle \mathbf{Q}_1, \mathbf{Q}_2, \mathbf{Q}_3 \rangle$, where $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are rank 4 quadrics. Let $\pi : T = \mathbf{Q}_1 \cap \mathbf{Q}_2 \cap \mathbf{Q}_3 \to S$ be the quotient map by the involution ι_T, which is an étale double covering of S and $E_1 = \pi^{-1}(e_1), E_2 = \pi^{-1}(e_2)$. Then we know that

$$E_1 = \mathbf{P}^3_1 \cap (\mathbf{Q}_2 \cap \mathbf{Q}_3), E_2 = \mathbf{P}^3_2 \cap (\mathbf{Q}_1 \cap \mathbf{Q}_3)$$
$$E_1 \cap E_2 = \mathbf{P}^2_{\ell_1, q_1} \cap (\mathbf{Q}_2 \cap \mathbf{Q}_3),$$

where E_1, E_2 are invariant non-singular elliptic quartic space curves and $\mathbf{P}^2_{\ell_1, q_1}$ invariant plane generated by the singular line $\ell_1 \subset \mathbf{Q}_1$ and a fixed point $q_1 \in Q_1$, where $\mathbf{Q}_1 = \langle \ell_1, Q_1 \rangle$. It is obvious that e_1, e_2 meet tangentially at a point $p_1 \in S$ if and only if E_1, E_2 meet tangentially at two points q_1', q_1'', where $\pi^{-1}(p_1) = q_1' \cup q_1''$. On the other hand, since $E_1 \cap E_2$ is the intersection of two plane conics $A_1 = \mathbf{P}^2_{\ell_1, q_1} \cap \mathbf{Q}_2$, $A_2 = \mathbf{P}^2_{\ell_1, q_1} \cap \mathbf{Q}_3$, we see that E_1, E_2 will meet tangentially at points q_1', q_1'' if and only if plane conics A_1, A_2 meet tangentially at the same two points q_1', q_1''.

PROPOSITION 3.1.. *We have the following equivalent relations $\mathcal{R}_1$:*

(1) *e_1, e_2 meet tangentially at a point p_1, that is,*

$$e_1 \cdot e_2 = 2\, p_1$$

(2) *E_1, E_2 meet tangentially at two points q_1', q_1'', that is,*

$$E_1 \cdot E_2 = A_1 \cdot A_2 = 2\,q_1' + 2\,q_1'' \text{ , where } \pi^{-1}(p_1) = q_1' \cup q_1''$$

(3) *If $\mathcal{L}_1$ is the pencil of quadrics generated by $\mathbf{Q}_2$ and $\mathbf{Q}_3$, then the pencil of conics $\mathcal{L}_1'$ which is the restriction of the pencil of quadrics $\mathcal{L}_1$ to the invariant plane $\mathbf{P}^2_{\ell_1,q_1}$ has the Segre symbol [(1,1),1],* [**7**].

(4) *The pencil of conics $\mathcal{L}_1'$ contains a double line, say $2\ell_1'$.*

(5) *There is a quadric $\mathbf{Q}_1' \in \mathcal{L}_1$ such that $\mathbf{Q}_1' \cdot \mathbf{P}^2_{\ell_1,q_1} = 2\ell_1'$. In other words, the invariant plane $\mathbf{P}^2_{\ell_1,q_1}$ meets the quadric $\mathbf{Q}_1'$ along a line ℓ_1' with the multiplicity two.*

(6) *For all quadrics $\mathbf{Q}$ in the pencil of quadrics generated by $\mathbf{Q}_1'$ and the rank 4 quadric $\mathbf{Q}_1$,*

$$\mathbf{Q} \cdot \mathbf{P}^2_{\ell_1,q_1} = 2\ell_1'$$

Proof. (1) $\iff$ (2) is obvious. (2) $\iff$ (3) follows from the classification of pencils of conics and their corresponding Segre symbols. It is also easy to check the rest of equivalent relations (3) $\iff$ (4) $\iff$ (5) $\iff$ (6).

Similarly, we get similar equivalent relations, say $\mathcal{R}_2$, for e_3, e_4 to meet tangentially at a point $p_2 \in S$ after replacing E_1, E_2 by $E_3 = \pi^{-1}(e_3), E_4 = \pi^{-1}(e_4)$ and $\mathbf{P}^2_{\ell_1,q_1}$ by $\mathbf{P}^2_{\ell_2,q_2}$ the invariant plane generated by the singular line ℓ_2 of $\mathbf{Q}_2$ and a fixed point $q_2 \in Q_2 \subset \mathbf{Q}_2$. Let $\pi^{-1}(p_2) = q_2' \cup q_2''$

Let us assume that the divisor $D = e_1 + e_2 + e_3 + e_4$ satisfies the equivalent conditions $\mathcal{R}_1$ and $\mathcal{R}_2$. Then for the divisor D to have the configuration in Figure 1, we need only one more requirement that

$$p_1 = p_2,$$

which is equivalent to saying that $q_1' = q_2'$, $q_1'' = q_2''$ or $q_1' = q_2''$, $q_1'' = q_2'$.

Let $N(\mathbf{P}^2_{\ell_1,q_1} \subset \mathbf{Q}_1, \mathbf{P}^2_{\ell_2,q_2} \subset \mathbf{Q}_2, \mathbf{Q}_3, \mathbf{Q}_1' \in \mathcal{L}_1, \mathbf{Q}_2' \in \mathcal{L}_2)$ be a net of quadrics in $\mathbf{P}^5$ from which we get four invariant non-singular elliptic space quartic curves E_1, E_2, E_3, E_4 of $T = \mathbf{Q}_1 \cap \mathbf{Q}_2 \cap \mathbf{Q}_3$, where $\mathbf{Q}_1, \mathbf{Q}_2$ are rank 4 quadrics and $\mathbf{Q}_1', \mathbf{Q}_2'$ quadrics belonging to the net N such that

$$\mathbf{Q}_1' \cdot \mathbf{P}^2_{\ell_1,q_1} = 2\ell_1', \quad \mathbf{Q}_2' \cdot \mathbf{P}^2_{\ell_2,q_2} = 2\ell_2' \tag{4}$$

for lines ℓ_1' and ℓ_2'. Then we have only the following three choices for the net of quadrics $N(\mathbf{P}^2_{\ell_1,q_1} \subset \mathbf{Q}_1, \mathbf{P}^2_{\ell_2,q_2} \subset \mathbf{Q}_2, \mathbf{Q}_3, \mathbf{Q}_1' \in \mathcal{L}_1, \mathbf{Q}_2' \in \mathcal{L}_2)$ with respect to four invariant non-singular ellitic curves E_1, E_2, E_3, E_4 of T :

$$(5)\quad \begin{array}{ll} (\alpha) & \mathbf{Q}_1 \neq \mathbf{Q}_2,\ \mathbf{Q}_1' = \mathbf{Q}_2' = \mathbf{Q}_3 \\ (\beta) & \mathbf{Q}_1 = \mathbf{Q}_2 = \mathbf{Q},\ \mathbf{P}^2_{\ell_1,q_1},\ \mathbf{P}^2_{\ell_2,q_2} \subset \mathbf{Q} \quad \text{for } q_1, q_2 \in Q \subset \mathbf{Q}, \\ & \text{where } q_1 \neq q_2 \text{ and } \ell_1 = \ell_2 = \ell \\ (\gamma) & \mathbf{Q}_1' = \mathbf{Q}_2 \text{ and } \mathbf{Q}_2' = \mathbf{Q}_1. \end{array}$$

Let N be the net of quadrics satisfying the case (α). Then to get a divisor D with the configuration in Figure 1, first of all, the identities in (4) have to hold true with $\ell_1' = \ell_2' = \ell'$. If it were true, then the invariant line ℓ' belongs to $T = \mathbf{Q}_1 \cap \mathbf{Q}_2 \cap \mathbf{Q}_3$ because $\ell' \subset \mathbf{P}^2_{\ell_1,q_1} \subset \mathbf{Q}_1$, $\ell' \subset \mathbf{P}^2_{\ell_2,q_2} \subset \mathbf{Q}_2$, $\ell' \subset \mathbf{Q}_3$. In particular, T contains two fixed points of ℓ' which is not necessarily rational double points. Thus T can not be birationally isomorphic to K3 surface with the fixed points free involution ι_T which is induced from the involution ι of $\mathbf{P}^5$.

Now let us examine the case (β). To have a divisor D with the configuration in Figure 1, we again have to have the identities (4) with $\ell_1' = \ell_2' = \ell'$ and moreover

$$E_1 \cdot E_2 = 2q_1 + 2q_2\,,\ E_3 \cdot E_4 = 2q_1 + 2q_2$$

for points q_1, q_2 of ℓ'. Then it forces $\ell' = \ell$, the singular line of $\mathbf{Q}$. Two double points q_1, q_2 are in the base space T, which are also points at which invariant elliptic curves E_1, E_2 and E_3, E_4 meet tangentially. If we blow up the base space T of the net of quadrics N at two points q_1, q_2 to get a smooth K3 surface, then the corresponding elliptic curves, which are the proper transform of E_1, E_2, E_3, E_4, do not meet tangentially any more.

Hence we conclude that from case (α) and (β), we can not get an Enriques surface S with a divisor D with the configuration in Figure 1. Thus it is obvious that the only possible choice for our purpose is the last one (γ). Notice that the case (γ) does not involve the third quadric $\mathbf{Q}_3$.

Employing the case (γ), we will show that we can get an Enriques surface S with the divisor D with the configuration in Figure 1.

Let $\mathbf{Q}_1, \mathbf{Q}_2, \mathbf{Q}_3$ be the generators of a net of quadrics N, where $\mathbf{Q}_1, \mathbf{Q}_2$

are rank 4 quadrics. We may choose the defining matrix of $\mathbf{Q}_1$ to be

$$\mathbf{Q}_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Let $\mathbf{Q}_2 = \left(\mathbf{U}^{-1}\right)^t \cdot \mathbf{Q} \cdot \mathbf{U}^{-1}$, which is equivalent to saying that the second quadric $\mathbf{Q}_2$ is the image of $\mathbf{Q}$ in $\mathbf{P}^5$ by the linear transformation $\mathbf{U}$,

$$\mathbf{Q} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{U} = \begin{pmatrix} \mathbf{A} & 0 \\ 0 & \mathbf{B} \end{pmatrix}$$

$$\mathbf{U}^{-1} = \begin{pmatrix} \mathbf{S} & 0 \\ 0 & \mathbf{T}, \end{pmatrix}$$

where $\mathbf{A} = (a_{ij}), \mathbf{B} = (b_{ij}), \mathbf{S} = (s_{ij})$ and $\mathbf{T} = (t_{ij})$ are 3×3 non-singular matrices.

To satisfy relations $\mathcal{R}_1$, $\mathcal{R}_2$ and the case (γ) of (5), we get the following equations (For the details, we refer to [**8**]):

$$(6) \qquad \begin{cases} t_{11} = 0 \;\&\; s_{11}s_{23} - s_{13}s_{21} = 0 \\ (s_{12}s_{23} - s_{13}s_{22})/\det(\mathbf{U}^{-1}) = -s_{13} \\ (s_{11}s_{22} - s_{12}s_{21})/\det(\mathbf{U}^{-1}) = s_{21} \\ t_{13}t_{21}/\det(\mathbf{U}^{-1}) = -t_{21} \\ t_{12}t_{31}/\det(\mathbf{U}^{-1}) = t_{31} \end{cases}$$

Hence our conclusion is as follows:

THEOREM 3.2.. *If we choose a net of quadrics N so that it satisfies the equations of* (6), *then from this net of quadrics N we get an Enriques surface S with a divisor D with the configuration in figure 1.*

We will present one very special net of quadrics N from which we could get an Enriques surface with a divisor D with the configuration in Figure 1. In equation (6) we set

$$s_{11} = s_{21}\,,\ s_{12} = s_{22} - 1\,,\ s_{13} = s_{23}\,,\ t_{11} = 0\,,\ t_{12} = 1\,,\ t_{13} = -1$$
$$s_{21} = 3\,,\ s_{33} = -1\,,\ s_{23} = 1\,,\ s_{31} = -4\,,\ t_{21} = 3\,,\ t_{32} = 1$$
$$t_{22} = 1\,,\ t_{31} = 2\,,\ t_{22} = 5\,,\ t_{32} = 2\,,\ t_{23} = 7\,,\ t_{33} = 4$$

Then it is easy to check that this choice of s's and t's satisfies the equation (6). We have the corresponding matrix $\mathbf{U}^{-1}$,

$$\mathbf{U}^{-1} = \begin{pmatrix} 3 & 4 & 1 & 0 & 0 & 0 \\ 3 & 5 & 1 & 0 & 0 & 0 \\ -4 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 3 & 1 & 7 \\ 0 & 0 & 0 & 2 & 1 & 4 \end{pmatrix}$$

Let $\mathbf{Q}_1, \mathbf{Q}_2$ be two rank 4 quadrics with the defining matrices

$$\mathbf{Q}_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\mathbf{Q}_2 = \begin{pmatrix} 18 & 27 & 6 & 0 & 0 & 0 \\ 27 & 40 & 9 & 0 & 0 & 0 \\ 6 & 9 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 3 & 2 & 6 \\ 0 & 0 & 0 & -3 & 6 & -14 \end{pmatrix}$$

And we take the third quadric $\mathbf{Q}_3$ with its defining matrix

$$\mathbf{Q}_3 = \begin{pmatrix} -5 & 3 & 2 & 0 & 0 & 0 \\ 3 & 1 & -2 & 0 & 0 & 0 \\ 2 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & -1 & 3 \\ 0 & 0 & 0 & -1 & 1 & 5 \\ 0 & 0 & 0 & 3 & 5 & -2 \end{pmatrix}$$

Note that $\mathbf{Q}_2 = \left(\mathbf{U}^{-1}\right)^t \cdot \mathbf{Q} \cdot \mathbf{U}^{-1}$, where

$$\mathbf{Q} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The relation $\mathbf{Q}_2 = \left(\mathbf{U}^{-1}\right)^t \cdot \mathbf{Q} \cdot \mathbf{U}^{-1}$ is equivalent to saying that the quadric $\mathbf{Q}_2$ is the image of $\mathbf{Q}$ by the linear transform $\mathbf{U}$. Three quadrics $\mathbf{Q}_1, \mathbf{Q}_2$ and $\mathbf{Q}_3$ define an Enriques net of quadrics N in $\mathbf{P}^5$. Then since this net of quadrics N is from a very special choice satisfying the equation (6), this special net of quadrics in $\mathbf{P}^5$ generates an Enriques surface S with a divisor D with the configuration in Figure 1.

THEOREM 3.3. *Let $\mathcal{N}$ be the space of Enriques nets of quadrics in $\mathbf{P}^5$, and $\mathcal{N}_1$ be the subvariety of $\mathcal{N}$ whose elements are nets of quadrics in $\mathbf{P}^5$ from which we get Enriques surfaces with a divisor D with the configuration in Figure 1. Then the codimension of $\mathcal{N}_1$ in $\mathcal{N}$ is 4.*

Proof. Let $N = \langle \mathbf{Q}_1, \mathbf{Q}_2, \mathbf{Q}_3 \rangle \in \mathcal{N}$, where $\mathbf{Q}_1, \mathbf{Q}_2$ are rank 4 quadrics in $\mathbf{P}^5$. Let $\mathbf{P}^2_1 \subset \mathbf{Q}_1$, $\mathbf{P}^2_2 \subset \mathbf{Q}_2$ be the invariant planes generated by the singular lines and fixed points. Then from the above computations, we see that the corresponding matrices of $\mathbf{P}^2_1 \cap \mathbf{Q}_2$ and $\mathbf{P}^2_2 \cap \mathbf{Q}_1$ are either

$$\begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{12} & a_{22} & 0 \\ 0 & 0 & b_{33} \end{pmatrix}$$

or

$$\begin{pmatrix} c_{11} & 0 & 0 \\ 0 & d_{22} & d_{23} \\ 0 & d_{23} & d_{33} \end{pmatrix}$$

Let $\mathcal{Q}_1 = \mathcal{Q}|_{\mathbf{P}^2_1}$, $\mathcal{Q}_2 = \mathcal{Q}|_{\mathbf{P}^2_2}$. Let $\mathcal{L} \subset \mathcal{Q}$ be the pencil of quadrics generated by $\mathbf{Q}_1$ and $\mathbf{Q}_2$, and

$$\mathcal{L}_1 = \mathcal{L}|_{\mathbf{P}^2_1} \subset \mathcal{Q}_1 \ , \ \mathcal{L}_2 = \mathcal{L}|_{\mathbf{P}^2_2} \subset \mathcal{Q}_2$$

pencils of conics that are the restriction of $\mathcal{L}$ to the invariant planes $\mathbf{P}^2_1$, $\mathbf{P}^2_2$ respectively. Then all conics of $\mathcal{Q}_1$ and $\mathcal{L}_1$ have only one of the above two defining matrices. Similarly for $\mathcal{Q}_2$ and $\mathcal{L}_2$. Let us assume that defining matrices of conics of $\mathcal{Q}_1$ are given by the first one. Thus $\mathcal{Q}_1$ is a projective three space $\mathbf{P}^3(a_{11}, a_{22}, a_{12}, b_{33})$. Let $\mathcal{S} \subset \mathcal{Q}$ be the space of singular quadrics, and $\mathcal{S}_i = \mathcal{S}|_{\mathbf{P}_i^2}$ for $i = 1, 2$. Then since $\mathcal{S}_1$ is the space of conics such that the determinants of their defining matrices are zero, that is,

$$(a_{11}a_{22} - a_{12}^2)b_{33} = 0,$$

$\mathcal{S}_1$ is a reducible cubic surface that is the union of a plane, $\alpha : b_{33} = 0$ and a cone which is the join of a plane conic $C : (a_{11}a_{22} - a_{12}^2) = 0 \,\&\, b_{33} = 0$ and a point $p = (0, 0, 0, 1)$ which corresponds to a corank 2 matrix

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Let $\mathcal{S}_i' \subset \mathcal{S}_i$ (i=1,2) be the space of conics with the corank 2. Then $\mathcal{S}_1'$ is composed of the plane conic C and a double point $p = (0, 0, 0, 1)$. Similarly for $\mathcal{S}_2'$. Thus it follows that $\mathbf{P}_1^2 \cap \mathbf{Q}_2$ and $\mathbf{P}_2^2 \cap \mathbf{Q}_1$ are double lines ℓ_1 and ℓ_2 if and only if

$$\mathcal{L}_i \cap \mathcal{S}_i' \neq \emptyset \quad \text{for } i = 1 \text{ and } 2$$

Obviously each of these equations gives a codimension one condition. For our purpose, we need one more condition which is $\ell_1 = \ell_2$. Since $\ell_i \subset \mathbf{P}_i^2\,(i = 1, 2)$ are invariant lines, the last one is a codimension two linear condition. Therefore, we need a codimension four condition altogether to get an Enriques surface with a divisor D with the configuration in Figure 1 from an Enriques net of quadrics in $\mathbf{P}^5$.

COROLLARY 3.4.. *The dimension $\mathcal{E}_1$ of the moduli space of Enriques surfaces S with a divisor D with the configuration in Figure 1 is 6.*

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