Honam Mathematical J. **17**(1995), No. 1,pp. 107–118

ASYMPTOTIC BEHAVIOR OF SINGULAR SOLUTIONS OF SEMILINEAR PARABOLIC EQUATIONS

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Abstract

We study the asymptotic behavior of nonnegative singular solutions of semilinear parabolic equations of the type

$$u_t = \Delta u - (u^q)_y - u^p$$

defined in the whole space $\mathbf{x} = (x, y) \in \mathbf{R}^{N-1} \times \mathbf{R}$ for $t > 0$, with initial data a Dirac mass, $\delta(\mathbf{x})$. The exponents q, p satisfy

$$1 < p < 1 + \frac{2q^*}{N+1}, \qquad 1 < q < \frac{N+1}{N-1},$$

where $q^* = max\{q, (N+1)/N\}$.

1. Introduction

In this paper we study the asymptotic behavior of singular solutions of nonnegative diffusion-convection equations with absorption of the form

$$\text{(F)} \qquad u_t = \Delta u - (u^q)_y - u^p$$

defined in the domain

$$Q = \{(\mathbf{x}, t) = (x, y, t) : (x, y) \in \mathbf{R}^{N-1} \times \mathbf{R}, t > 0\}$$

with initial data a Dirac mass $\delta(\mathbf{x})$.

Received May 4, 1995.

This work was supported by Nondirect Research Fund, Korea Research Foundation, 1994, KOSEF-GARC, BSRI-94-1426 and MRC-CNU.

Baek and Kwak (see [BK]) showed that there exists a unique nonnegative singular solution $u(\mathbf{x},t)$ of (F) such that $u(\mathbf{x},t) \to \delta(\mathbf{x})$ as $t \to 0$ in the sense of measures, that is,

$$\lim_{t\to 0} \int_{\mathbf{R}^N} u(\mathbf{x},t)\phi(\mathbf{x})d\mathbf{x} = \phi(0)$$

for all continuous and bounded function ϕ on $\mathbf{R}^N$ if and only if $1 < p < 1+2q^*/(N+1)$ and $1 < q < (N+1)/(N-1)$. Here $q^* = max\{q, (N+1)/N\}$.

The behavior of solutions of (F) will be completely decided by that of the following equations :

$$u_t = \Delta u \tag{1.1}$$

$$u_t = \Delta u - u^p \tag{1.2}$$

$$u_t = \Delta_x u - (u^q)_y \tag{1.3}$$

$$u_t = \Delta_x u - (u^q)_y - u^p, \tag{1.4}$$

where Δ_x denotes the Laplace operator acting only on the variable x. The singular solution of (1.1) is the standard heat kernel

$$G(\mathbf{x},t) = (4\pi t)^{-\frac{N}{2}} e^{-\frac{|\mathbf{x}|^2}{4t}}.$$

We also recall that (1.2) has a unique very singular solution $W(\mathbf{x},t)$ which has a stronger singularity at 0, i.e., such that

$$\lim_{t\to 0} \int W(\mathbf{x},t)d\mathbf{x} = +\infty,$$

see [BPT]. The existence of singular solutions of (1.3) is proved in [BK] and [EVZ]. The existence of singular solutions of (1.4) will be discussed in other space.

We denote by $\|\cdot\|_r$ the usual norm of $L^r(\mathbf{R}^N)$, $1 \le r \le \infty$, and we prove that

THEOREM A. *Suppose* $(N+2)/N < p < 1 + 2q/(N+1)$ *and* $(N+1)/N < q < (N+1)/(N-1)$. *Then the singular solution of* (F) *satisfies*

$$\lim_{t\to\infty} t^{\frac{N}{2}(1-\frac{1}{r})}\|u(\mathbf{x},t) - G(\mathbf{x},t)\|_r = 0.$$

Let us denote by $C(\mathbf{x},t)$ the singular solution of (1.3) and then we prove

THEOREM B. *Suppose* $1 + 2q/(N+1) < p < (N+2)/N$ *and* $1 < q < (N+1)/N$. *Then the singular solution of* (F) *satisfies*

$$\lim_{t\to\infty} t^{\frac{N+1}{2q}(1-\frac{1}{r})}\|u(\mathbf{x},t) - C(\mathbf{x},t)\|_r = 0.$$

We also prove

THEOREM C. *Let* $q_* = min\{q,(N+1)/N\}$. *If* $1 < p < 1+2q_*/(N+1)$, $(1+p)/2 < q < (N+1)/(N-1)$ *and* $u(\mathbf{x},t)$ *is the singular solution of* (F), *then*

$$t^{\frac{1}{p-1}}|u(\mathbf{x},t) - W(\mathbf{x},t)| \to 0 \qquad \text{as} \quad t\to\infty,$$

uniformly on the sets $\{\mathbf{x}\in\mathbf{R}^N : |\mathbf{x}| \le at^{\frac{1}{2}}\}$, $\forall a > 0$.

For the proof of theorems, we introduce a similarity transformation

$$u_\lambda(x,y,t) = \lambda^\alpha u(\lambda x, \lambda^\beta y, \lambda^2 t)$$

with appropriate choice of constants α and β.

By applying compactness arguments, we deduce that u_λ converges to one of the singular solutions $G(\mathbf{x},t)$, $C(\mathbf{x},t)$, $W(\mathbf{x},t)$ as $\lambda\to\infty$. As converting to the behavior as $t\to\infty$, we obtain Theorem A, B and C.

2. Proof of Theorem A

For the singular solution u of (F), we define

$$u_\lambda(\mathbf{x},t) = \lambda^N u(\lambda\mathbf{x}, \lambda^2 t).$$

Then u_λ satisfies the equation

$$u_{\lambda,t} = \Delta u_\lambda - \lambda^{N+1-Nq}(u_\lambda^q)_y - \lambda^{N+2-Np}u_\lambda^p. \tag{2.1}$$

For λ very large, we may view (2.1) as a small perturbation of the linear heat equation (1.1) since λ^{N+1-Nq} and λ^{N+2-Np} become sufficiently small.

Now note that

$$\begin{aligned}|u_\lambda(\mathbf{x},1) - G(\mathbf{x},1)| &= |\lambda^N u(\lambda\mathbf{x},\lambda^2) - G(\mathbf{x},1)|\\ &= |\tau^{N/2} u(\tilde{\mathbf{x}},\tau) - G(\frac{\tilde{\mathbf{x}}}{\sqrt{\tau}},1)|\\ &= \tau^{N/2}|u(\tilde{\mathbf{x}},\tau) - G(\tilde{\mathbf{x}},\tau)|\end{aligned}$$

where $\tilde{\mathbf{x}} = \lambda\mathbf{x}$ and $\lambda^2 = \tau$.

Therefore, if $u_\lambda(\mathbf{x},1)$ converges to $G(\mathbf{x},1)$ as $\lambda \to \infty$, then we obtain $u(\mathbf{x},t) \to G(\mathbf{x},t)$ in the same norm. In particular, if

$$\|u_\lambda(\mathbf{x},1) - G(\mathbf{x},1)\|_r \to 0 \qquad \text{as} \quad \lambda \to \infty$$

then

$$\tau^{\frac{N}{2}(1-\frac{1}{r})}\|u(\mathbf{x},\tau) - G(\mathbf{x},\tau)\|_r \to 0 \qquad \text{as} \quad \tau \to \infty.$$

For the completion of proof we first recall two basic estimates for singular solution of (F).

$$0 \le u(\mathbf{x},t) \le C(t^{-\frac{N}{2}} + t^{(1-Nq)/2}), \qquad \forall t > 0. \tag{2.2}$$

$$0 \le u(\mathbf{x},t) \le C(q,N)t^{-(N+1)/2q}, \qquad \forall t > 0. \tag{2.3}$$

The former is proved in [EZ] and the latter is proved in [BK] and in [EVZ].

We also need the following lemma.

LEMMA 2.1. *For every $\tau > 0$, there exists a constant C_τ such that*

$$\|\nabla u_\lambda(t)\|_1 \le C_\tau(t-\tau)^{-1/2}$$

for every $t > \tau$ and for all $\lambda \ge 1$.

Proof. In view of equation (2.1), u_λ satisfies

$$\begin{aligned}u_\lambda(t+\tau) =& G(\mathbf{x},t) * u_\lambda(\tau)\\ &-\lambda^{N+1-Nq}\int_0^t (G(\mathbf{x},t-s) * (u_\lambda{}^q(s+\tau))_y)\,ds\\ &-\lambda^{N+2-Np}\int_0^t (G(\mathbf{x},t-s) * u^p(s+\tau))\,ds,\end{aligned}$$

where $*$ denotes the convolution in $\mathbf{R}^N$. Differentiating, we get

$$\begin{aligned}\nabla u_\lambda(t+\tau) =& \nabla G(t) * u_\lambda(\tau) \\ & -\lambda^{N+1-Nq}\int_0^t \nabla G(t-s) * (u_\lambda{}^q(s+\tau))_y ds \\ & -\lambda^{N+2-Np}\int_0^t \nabla G(t-s) * u^p(s+\tau)ds,\end{aligned}$$

We take L^1-norm in space variable and use (2.3) to obtain

$$\|\nabla u_\lambda(t+\tau)\|_1 \leq C_1(\tau)t^{-\frac{1}{2}} + C_2(\tau)\int_0^t (t-s)^{-1/2}\|\nabla u_\lambda(s+\tau)\|_1 ds$$

for $t > 0$ and $\lambda \geq 1$.

By applying the Gronwall's inequality we obtain the Lemma.

Equation (2.1), (2.2), (2.3) and Lemma 2.1 imply that

(i) $\{u_\lambda^{q+r}\}_\lambda$ is uniformly bounded in $L^\infty((\tau,\infty) : W^{1,1}(\mathbf{R}^N)))$ for every $r > 0$ and $\tau > 0$.

(ii) $\{u_{\lambda,t}\}_\lambda$ is uniformly bounded in $L^2_{loc}((0,\infty) : H^{-s}(\Omega))$ for some $s > 0$ and every bounded domain Ω of $\mathbf{R}^N$.

(iii) $\{u_\lambda\}$ is uniformly bounded in $L^\infty_{loc}((0,\infty) : L^2_{loc}(\mathbf{R}^N))$.

Taking into account that $L^2(\Omega)$ is compactly embeded in $H^{-\epsilon}(\Omega)$ for every ϵ, and that $H^{-\epsilon}(\Omega)$ is continuously embeded in $H^{-s}(\Omega)$ for every $s > \epsilon$, combining (ii) and (iii) we deduce that

(iv) $\{u_\lambda\}$ is relatively compact in $C([t_1,t_2] : H^{-\epsilon}(\Omega))$ for some $\epsilon > 0$.

Here for some sequence $\lambda_n \to \infty$, we may assent that (2.4) $u_{\lambda_n} \to U$ in $C([t_1,t_2] : H^{-\epsilon}(\Omega))$ for every bounded domain Ω. As a consequence of (i), we conclude that $u_\lambda(t)$ is relatively compact in $L^r_{loc}(\mathbf{R}^N)$ for every $1 \leq r < \infty$ and $t > 0$. In view of (2.4), we get $u_{\lambda_n} \to U$ in $L^r_{loc}(\mathbf{R}^N)$. And U is a solution of the heat equation in the sense of distribution.

We now check the initial condition of U. We multiply equation (2.1) by a test function $\phi(\mathbf{x}) \in C_0^\infty(\mathbf{R}^N)$ and integrate over $\mathbf{R}^N \times (0,t)$. Then

$$\begin{aligned}&\left|\int u_\lambda(\mathbf{x},t)\phi(\mathbf{x})d\mathbf{x} - \int u_\lambda(\mathbf{x},0)\phi(\mathbf{x})d\mathbf{x}\right| \\ =& \left|\int_0^t\int u_\lambda(\mathbf{x},t)\Delta\phi(\mathbf{x})d\mathbf{x}ds + I_1(\lambda,t) - I_2(\lambda,t)\right| \\ \leq& \|\Delta\phi\|_{L^\infty}t + |I_1(\lambda,t)| + |I_2(\lambda,t)|,\end{aligned}$$

where

$$I_1(\lambda,t) = \lambda^{N+1-Nq}\int_0^t\int u_\lambda^q(\mathbf{x},s)\phi_y(\mathbf{x})d\mathbf{x}ds,$$
$$I_2(\lambda,t) = \lambda^{N+1-Np}\int_0^t\int u_\lambda^p(\mathbf{x},s)\phi(\mathbf{x})d\mathbf{x}ds.$$

Since $\int u_\lambda dx \le 1$, from (2.3) we obtain

$$|I_1(\lambda,t)| \le C(q,N)\|\phi_y\|_{L^\infty}\lambda^{\frac{N+1-Nq}{q}}\int_0^t s^{-\frac{(N+1)(q-1)}{2q}}ds,$$
$$|I_2(\lambda,t)| \le C(q,N)\|\phi\|_{L^\infty}\lambda^{\frac{2q-(N+1)(p-1)}{q}}\int_0^t s^{-\frac{(N+1)(p-1)}{2q}}ds.$$

If $p < 1+2q/(N+1)$ and $q > (N+1)/N$, then both I_1 and I_2 tend to 0 as $\lambda \to \infty$ and $t \to 0$. Since $\int u_\lambda(\mathbf{x},0)d\mathbf{x} = \phi(0)$, we see that $\lim_{t\to 0} U(\mathbf{x},t) = \delta(\mathbf{x})$.

According to the uniqueness of the singular solution of the heat equation, we see that U is infact the heat kernel $G(\mathbf{x},t)$.

We have shown that u_λ converges locally in $L^r(\mathbf{R}^N)$. We now prove that u_λ converges to G in $L^r(\mathbf{R}^N)$. Fix a positive time, say $t=1$. Then given $\epsilon > 0$ and sufficiently large R satisfying $\int_{|\mathbf{x}|>R} G(\mathbf{x},t)d\mathbf{x} \le \epsilon$, there exists λ_0 such that

$$\int_{|\mathbf{x}|<R} |u_\lambda(\mathbf{x},1) - G(\mathbf{x},1)|d\mathbf{x} \le \epsilon \qquad \text{for} \quad \lambda > \lambda_0.$$

Since $\int u_\lambda(\mathbf{x},1)d\mathbf{x} \le \int G(\mathbf{x},1)d\mathbf{x} = 1$, we obtain $\int_{|\mathbf{x}|<R} u_\lambda d\mathbf{x} \ge 1-2\epsilon$ and $\int_{|\mathbf{x}|>R} u_\lambda(\mathbf{x},1)d\mathbf{x} \le 2\epsilon$. These imply that

$$\int_{\mathbf{R}^N} |u_\lambda(\mathbf{x},1) - G(\mathbf{x},1)|d\mathbf{x} \le 4\epsilon.$$

Note that in view of (2.2), $u_\lambda(\mathbf{x},1)$ is uniformly bounded for $\lambda \ge 1$. Hence we get

$$\|u_\lambda(\mathbf{x},t) - G(\mathbf{x},t)\|_r \le \|u_\lambda - G\|_\infty^{(r-1)/r}\|u_\lambda - G\|_{L^1}^{1/r},$$

which tends to 0 as $\lambda \to \infty$.

3. Proof of Theorem B

For the singular solution u of (F), we now consider

$$u_\lambda(x, y, t) = \lambda^{(N+1)/q} u(\lambda x, \lambda^\alpha y, \lambda^2 t),$$

where $\alpha = (N+1+q-Nq)/q$. Then u_λ satisfy the equation

$$(3.1)\quad u_{\lambda,t} = \Delta_x u_\lambda + \lambda^{2(Nq-N-1)/q} u_{\lambda,yy} - (u_\lambda^q)_y - \lambda^{(-Np-p+N+1+2q)/q} u_\lambda^p.$$

For λ very large, we may view (3.1) as a small perturbation of (1.3). Since the solution $C(\mathbf{x}, t)$ of (1.3) is scaling invariant under the above transformation, it is enough to show that $u_\lambda(\mathbf{x}, 1)$ converges to $C(\mathbf{x}, 1)$.

From the estimate (2.3), $u_\lambda(\mathbf{x}, t)$ is uniformly bounded in $L^\infty(\mathbf{R}^N \times (\tau, \infty))$ for any $\tau > 0$ and we may extract a subsequence $\{u_{\lambda_j}\}_{j=1}^\infty$ which converges in the weak $*$ topology of L^∞. By applying the compensated compactness argument (see [E] and [T], Theorem 2.6), we may conclude that along such a solution

$$u_{\lambda_j} \to U \qquad \text{in} \quad L^r_{loc}(Q) \qquad \forall 1 \le r < \infty,$$

where U is an entrophy solution of the reduced equation (1.3). (See [EVZ])

In order to check the initial condition, it is enough to show that $I_3(\lambda, t)$ and $I_4(\lambda, t)$ tend to 0 as $\lambda \to \infty$ and $t \to 0$, where

$$I_3(\lambda, t) = \int_0^t \int_{\mathbf{R}^N} u_\lambda^q(\mathbf{x}, s)\phi_y(\mathbf{x}) d\mathbf{x} ds,$$

$$I_4(\lambda, t) = \lambda^{(-Np-p+N+1+2q)/q} \int_0^t \int_{\mathbf{R}^N} u_\lambda^p(\mathbf{x}, s)\phi(\mathbf{x}) d\mathbf{x} ds$$

for any $\phi(\mathbf{x}) \in C_0^\infty(\mathbf{R}^N)$. This follows from the following estimates

$$|I_3(\lambda, t)| \le C(q, N)\|\phi_y\|_{L^\infty} \int_0^t s^{-(N+1)(q-1)/(2q)} ds,$$

$$|I_4(\lambda, t)| \le C(q, N)\|\phi\|_{L^\infty} \lambda^{(-Np-p+N+1+2q)/q} \int_0^t s^{-(N+1)(p-1)/(2q)} ds.$$

Note that $1 + (2q)/(N+1) < p$ and $q < (N+1)/(N-1)$. Hence we obtain $\lim_{t\to 0} U(\mathbf{x}, t) = \delta(\mathbf{x})$.

According to the uniqueness result of the singular solution of (1.3), we may conclude that $U(\mathbf{x}, t) = C(\mathbf{x}, t)$. The proof of L^r-convergence of $u_\lambda(\mathbf{x}, 1)$ to $C(\mathbf{x}, 1)$ is similar to the proof of Theorem A.

4. Proof of Theorem C

For the proof we need a priori estimates in terms of space variables as well as time variables.

Let $u(x,y,t)$ be the singular solution of (F), then

$$0 \le u(x,y,t) \le (p-1)^{-\frac{1}{p-1}} t^{-\frac{1}{p-1}} \tag{4.1}$$

holds since the right member is a supersolution of (F). It is also easy to see that if we choose $M > 0$ so that $M^p \ge \frac{4pM}{(p-1)^2} + \frac{M}{p-1}$, then $\frac{M}{(|x|^2+t)^{1/(p-1)}}$ is a supersolution and

$$0 \le u(x,y,t) \le \frac{M}{(|x|^2+t)^{1/(p-1)}}. \tag{4.2}$$

For y-variable, when $y \le 0$, if we choose $L > 0$ so that $L^p \ge \frac{2L(p+1)}{(p-1)^2}$, then the Comparison Principle yields

$$0 \le u(x,y,t) \le \frac{L}{|y|^{2/(p-1)}}. \tag{4.3}$$

Now for $y > 0$, let $z(x,y,t) = u(x, y + h(t), t)$, then z satisfies

$$z_t = \Delta z + (h'(t) - qu^{q-1}) z_y - z^p.$$

We take $h(t)$ so that $h'(t) \ge qu^{q-1}$. For example, let

$$h'(t) = q(p-1)^{-(q-1)/(p-1)} t^{-(q-1)/(p-1)}$$

and

$$h(t) = \begin{cases} q(p-1)^{-\frac{q-1}{p-1}} \frac{p-1}{p-q} t^{\frac{p-q}{p-1}}, & \text{for } p \ne q \\ q(p-1)^{-\frac{q-1}{p-1}} \ln t, & \text{for } p = q. \end{cases}$$

Applying the Comparison Principle again, we obtain

$$0 \le z(x,y,t) = u(x, y+h(t), t) \le \frac{L}{|y|^{2/(p-1)}}, \quad y > 0 \tag{4.4}$$

as before (see (4.3)). Here $h'(t)$ and $u(x,y,t)$ become singular as $t \to 0$ but taking smooth initial data approximating $\delta(\mathbf{x})$, we first obtain estimates similar to (4.4) and we get (4.4) in the limit. From (4.4) we obtain

$$0 \le u(x,y,t) \le \frac{L}{([y-h(t)]^+)^{2/(p-1)}}, \quad y > 0. \tag{4.5}$$

Here $[x]^+ = \max\{0, x\}$.

We now turn to the proof of Theorem C.

Let $u_\lambda = \lambda^{2/(p-1)} u(\lambda x, \lambda y, \lambda^2 t)$, then u_λ satisfies

$$\begin{aligned} u_{\lambda,t} &= \Delta u_\lambda - \lambda^{\frac{2}{p-1}+1-\frac{2q}{p-1}} (u_\lambda^q)_y - u_\lambda^p \\ u_\lambda(\mathbf{x}, 0) &= \lambda^{\frac{2}{p-1}-N} \delta(\mathbf{x}). \end{aligned} \tag{4.6}$$

Assume $\frac{2}{p-1} + 1 - \frac{2q}{p-1} < 0$, that is, $2q > p+1$, then it is easy to see that $\{u_\lambda\}$ are uniformly bounded in every compact subset of $\overline{Q}\ \{(0,0)\}$ and $\{\nabla u_\lambda\}$ are uniformly Holder continuous in every compact set of Q. Hence there exists a subsequence $\{u_{\lambda_j}\}$ and function $U \in C(Q)$ such that

$$\begin{aligned} &u_{\lambda_j}(\mathbf{x}, t) \to U(\mathbf{x}, t), \\ &\nabla u_{\lambda_j}(\mathbf{x}, t) \to \nabla U(\mathbf{x}, t) \quad \text{as} \quad \lambda_j \to \infty \end{aligned}$$

uniformly on every compact subset of Q. Clearly U satisfies (1.2) in the sense of distribution and becomes a classical solution in Q from the standard regularity theory.

In order to check the initial condition, let $\phi_i(\geq 0) \in C_0^\infty(\mathbf{R}^N)$, $i = 1, 2, 3$ and

$$\begin{aligned} &\mathrm{supp}\phi_1 \subset \{(x, y) \in \mathbf{R}^N : x \neq 0\}, \\ &\mathrm{supp}\phi_2 \subset \{(x, y) \in \mathbf{R}^N : y < 0\}, \\ &\mathrm{supp}\phi_3 \subset \{(x, y) \in \mathbf{R}^N : y > 0\}. \end{aligned}$$

We mutiply these test functions to (4.6) and integrate to obtain

$$\begin{aligned} &\int u_\lambda(x, y, t)\phi_i(x, y)dxdy - \int u_\lambda(x, y, 0)\phi_i(x, y)dxdy \\ &\quad = \int_0^t \int u_\lambda(x, y, t)\Delta\phi_i(x, y)dxdydt \\ &\qquad + \lambda^{\frac{2}{p-1}+1-\frac{2q}{p-1}} \int_0^t \int u_\lambda^q(x, y, t)\phi_{iy}(x, y)dxdydt \\ &\qquad - \int_0^t \int u_\lambda^p(x, y, t)\phi_i(x, y)dxdydt. \end{aligned}$$

Since the second term on the left side becomes 0 and the last term on the right side is negative, we have that

$$\begin{aligned}\int u_\lambda(x,y,t)\phi_i(x,y)dxdy &- \int u_\lambda(x,y,0)\phi_i(x,y)dxdy \\ &\leq \int_0^t \int_{\text{supp}\phi_i} u_\lambda(x,y,t)\|\Delta\phi_i\|_{L^\infty} dxdydt \\ &\quad + \lambda^{\frac{2}{p-1}+1-\frac{2q}{p-1}} \int_0^t \int_{\text{supp}\phi_i} u_\lambda^q(x,y,t)\|\phi_{iy}\|_{L^\infty} dxdydt.\end{aligned}$$

From (4.2) and (4.3), u_λ, u_λ^q are integrable over $(0,t)\times \text{supp}\phi_i$, $i=1,2$. Thus taking $\lambda\to\infty$ and $t\to 0$ we obtain

$$\lim_{t\to 0}\int U(x,y,t)\phi_i(x,y)dxdy = 0 \tag{4.7}$$

for $i=1,2$. On $\text{supp}\phi_3$, from (4.5)

$$\begin{aligned}u_\lambda(x,y,t) &= \lambda^{\frac{2}{p-1}}u(\lambda x,\lambda y,\lambda^2 t) \\ &\leq \frac{\lambda^{2/(p-1)}L}{([\lambda y - h(\lambda^2 t)]^+)^{2/(p-1)}}.\end{aligned}$$

For $p<q$, $h(\lambda^2 t)<0$ and $u_\lambda(x,y,t)\leq \frac{L}{|y|^{2/(p-1)}}$.

For $p=q$, $1/\lambda h(\lambda^2 t) = 1/\lambda \ln(\lambda^2 t)$, which goes to 0 as $\lambda\to\infty$ and $t\to 0$.

For $q<p<2q-1$,

$$\frac{1}{\lambda}h(\lambda^2 t) = q(p-1)^{-\frac{q-1}{p-1}}\frac{p-1}{p-q}\lambda^{-1+\frac{2(p-q)}{p-1}}t^{\frac{p-q}{p-1}},$$

which goes to 0 as $\lambda\to\infty$ and $t\to 0$. Hence we see that for sufficiently large λ and small t, u_λ and u_λ^q are uniformly integrable over $(0,t)\times\text{supp}\phi_3$ and

$$\lim_{t\to 0}\int U(x,y,t)\phi_3(x,y)dxdy = 0. \tag{4.8}$$

From (4.7), (4.8), we may conclude that

$$\lim_{t\to 0}\int U(x,y,t)\phi(x,y)dxdy = 0 \qquad \forall\phi\in C_0^\infty(\mathbf{R}^N - \{0\}). \tag{4.9}$$

Finally for any $M > 0$, consider the solution v_λ of

$$\begin{aligned} v_{\lambda,t} &= \Delta v_\lambda - \lambda^{\frac{2}{p-1}+1-\frac{2q}{p-1}}(v_\lambda^q)_y - v_\lambda^p \\ v_\lambda(\mathbf{x},0) &= M\delta(\mathbf{x}). \end{aligned} \tag{4.10}$$

For all sufficiently large λ, $\lambda^{\frac{2}{p-1}-N} \geq M$ and from the Comparison Principle we get $0 \leq v_\lambda(\mathbf{x},t) \leq u_\lambda(\mathbf{x},t)$. It is easy to see that $\{v_\lambda\}$ converges to a singular solution $P_M(\mathbf{x},t)$ of (1.2) with total mass M. Hence we obtain $0 \leq P_M(\mathbf{x},t) \leq U(\mathbf{x},t)$. In particular

$$M = \lim_{t\to 0}\int P_M(\mathbf{x},t)d\mathbf{x} \leq \lim_{t\to 0}\int U(\mathbf{x},t)d\mathbf{x}.$$

This shows that

$$\lim_{t\to 0}\int U(\mathbf{x},t)d\mathbf{x} = \infty. \tag{4.11}$$

From (4.9), (4.11) and the uniqueness result we conclude that $U(\mathbf{x},t)$ is in fact the very singular solution of (1.2).(See [O], [KPV])

5. Final Remarks

The case $1 < p < 1 + (2q_*)/(N+1)$ and $1 < q < (1+p)/2$ is not considered here. Recall that $q_* = \min\{q, (N+1)/N\}$. We only presume that the singular solution of (F) behaves like a very singular solution of (1.4). But as far as we know, no research has been made on the singular solution of (1.4). Hence we have to make a little more efforts for the proof, which will be postponed to the forthcoming paper.

The borderline cases are not considered neither here. We believe that those solutions have self-similar profiles and we leave these cases to the interested reader.(See [EZ])

References

[E] L.C. Evans, *Weak convergence methods for nonlinear partial differential equations*, Conference board of the mathematical sciences, Regional conference series in Mathematics 74, A.M.S., Providence, 1988.

[PW] M.H. Protter and H.F. Weinberger, *Maximum principles in differential equations*, Springer-Verlag, New York, 1984.

[T] L. Tartar, *Compensated compactness and applications to partial differential equations*, Research notes in Mathematics 39, 1979.

[BK] J.S. Baek and M. Kwak, *Singular solutions of semilinear parabolic equations in several space dimensions*, Preprint series in GARC-SNU 94-36 (1994).

[EZ] M. Escobedo and E. Zuazua, *Large time behavior for convection diffusion equations in* $\mathbf{R}^N$, Jour. of Functional Analysis **100** (1991), 119-161.

[EVZ] M. Escobedo, J.L. Vazquez, and E. Zuazua, *A diffusion-convection equation in several space dimensions*, Indiana Univ. Math. Jour. **42 No. 4** (1993), 1413-1440.

[O] L. Oswald, *Isolated positive singularities for a nonlinear heat equation*, Houston J. of Math. **14, No. 4** (1988), 543-572.

[KPV] S.Kamin, L.A. Peletier, and J.L. Vazquez, *Classification of singular solutions of a nonlinear heat equation*, Duke Math. Jour. **58, No. 3** (1989), 601-615.