ON CERTAIN AREA FUNCTIONS ASSOCIATED WITH APPROACH REGIONS

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1. Introduction

In this paper we first define a group of homogeneous type $G$, which is a more general setting than $\mathbb{R}^n$, and we also consider the space $G \times (0, \infty)$, which is a kind of generalized upper half-space over $G$. Then we shall assume that to each boundary point $x \in G$, there is associated an approach region $\Gamma_\alpha(x) \subset G \times (0, \infty)$. Let $f$ be a function defined on $G \times (0, \infty)$. For $x \in G$ and $\alpha > 0$, we define an area function $S_\alpha(f)$ associated with $\Gamma_\alpha(x)$, by

$$S_\alpha(f)(x) = \left( \int_{\Gamma_\alpha(x)} |f(y,t)|^2 \frac{d\mu(y)dt}{tn+1} \right)^{1/2},$$

where $\mu$ denotes the Borel measure on $G$. For simplicity, we put $S(f) = S_1(f)$. The purpose of this paper is to study inequalities for the $L^p$ norms of area functions $S_\alpha(f)$ and $S(f)$ for $\alpha > 1$; more precisely, let $0 < p < \infty$ and $\alpha > 1$, then there is a constant $C$ such that

$$\|S_\alpha(f)\|_{L^p(d\mu)} \leq C \|S(f)\|_{L^p(d\mu)}.$$

Throughout this paper we shall use the letter $C$ to denote a constant which need not be the same at each occurrence.

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2. Preliminaries

Let $G$ be a topological group. Assume that $d$ is a pseudo-distance on $G$, i.e., a nonnegative function defined on $G \times G$ with properties

(i) $d(x, x) = 0; d(x, y) > 0$ if $x \neq y$,
(ii) $d(x, y) = d(y, x)$, and
(iii) there is a constant $K$ such that

$$d(x, z) \leq K[d(x, y) + d(y, z)] \quad \text{for all} \ x, y, z \in G.$$

Assume also that

(a) the balls $B(x, \rho) = \{y \in G : d(x, y) < \rho\}, \ \rho > 0$, form a basis of open neighborhoods at $x \in G$,

and that $\mu$ is a Borel measure on $G$, and

(b) there is a constant $A$ such that

$$0 < \mu(B(x, 2\rho)) \leq A\mu(B(x, \rho)) < \infty \quad \text{for all} \ x \in G, \ \rho > 0.$$

Assume further that $\mu$ is left-invariant:

(c) $\mu(xE) = \mu(E)$ for $x \in G$, measurable $E \subset G$, and
(d) $\mu(E^{-1}) = \mu(E)$,

and that $d$ is left-invariant:

(e) $xB(y, \rho) = B(xy, \rho)$ for all $x, y \in G, \ \rho > 0$.

Then we call $(G, d, \mu)$ a group of homogeneous type. Let $(G, d, \mu)$ be a group of homogeneous type and $\rho > 0$. Then an automorphism $\delta_\rho$ of $G$ is called a dilation of $G$ if there is a positive integer $n$ such that

$$\mu(\delta_\rho(E)) = \rho^n \mu(E) \quad \text{(1)}$$

for any measurable $E \subset G$, and in particular,

$$\mu(\delta_\rho(B(\epsilon, 1))) = \mu(B(\epsilon, \rho)) = C_n \rho^n, \quad \text{(2)}$$

where $C_n$ denotes the volume of the unit ball $B(\epsilon, 1)$, and $\epsilon$ denotes the identity element of $G$. Frequently, we shall write $\rho x$ instead of $\delta_\rho x$. 
for $\rho > 0$ and $x \in G$. For details see [6]. A. Korányi and S. Vági [3] studied that $d$ is left-invariant if and only if

$$d(x, y) = |x^{-1}y|,$$

where $| \cdot |$ is a nonnegative function on $G$ with properties

(i) $|x| = 0$ if and only if $x = e$,
(ii) there is a constant $K$ such that $|xy| \leq K(|x| + |y|)$, and
(iii) $|x^{-1}| = |x|$.

For $x, y \in G$ and $\rho > 0$, the set

$$B(x, \rho) = \{ y \in G : |x^{-1}y| < \rho \}$$

is called the ball centered at $x \in G$ with radius $\rho$. Now consider the space $G \times (0, \infty)$, which is a kind of generalized upper half-space over $G$. We then introduce the analogue of nontangential or conical region. For $x \in G$ and $\alpha > 0$, set

$$\Gamma_\alpha(x) = \{(y, t) \in G \times (0, \infty) : |x^{-1}y| < \alpha t\}.$$

For simplicity, we put $\Gamma(x) = \Gamma_1(x)$. For any closed subset $F \subset G$ and $\alpha > 0$, set

$$R_\alpha(F) = \bigcup_{x \in F} \Gamma_\alpha(x).$$

Then the tent over an open subset $O = ^c F$ of $G$, denoted by $T(O)$, is given as

$$T(O) = ^c R_1(F).$$

We define an area function associated with an approach region as follows. Let $f$ be a function defined on $G \times (0, \infty)$. For $x \in G$ and $\alpha > 0$, set

$$S_\alpha(f)(x) = \left( \int_{\Gamma_\alpha(x)} |f(y, t)|^2 \frac{d\mu(y)dt}{t^{n+1}} \right)^{1/2}.$$

Let $f$ be a locally integrable function on $G$. For $x \in G$, we define

$$M(f)(x) = \sup_{x \in B} \frac{1}{\mu(B)} \int_B |f(y)| d\mu(y),$$
where the supremum is taken over all balls $B$ containing $x$. Then $M(f)$ is called the \textit{Hardy-Littlewood maximal function} of $f$. We need the notion of points of density. Suppose $F$ is a closed subset of $G$ and $\gamma$ is a fixed parameter, $0 < \gamma < 1$. Then we say that a point $x \in G$ has \textit{global $\gamma$-density} with respect to $F$, if

$$\frac{\mu(F \cap B)}{\mu(B)} \geq \gamma$$

for all balls $B$ centered at $x$ in $G$. Let $F^*$ be the set of points of global $\gamma$-density with respect to $F$; then $F^*$ is closed, $F^* \subset F$, and

$${\mathcal{F}}^* = \{ x \in G : M(\chi_{\mathcal{F}})(x) > 1 - \gamma \},$$

where $\chi_{\mathcal{F}}$ is the characteristic function of the open set $F$.

3. Main result

We state the four lemmas we need.

**Lemma 1** [6]. Assume $F$ is a closed subset of $G$. Then there is a constant $C$ such that

$$\mu({\mathcal{F}}^*) \leq C \mu({\mathcal{F}}),$$

where $F^*$ is the set of points of global $\gamma$-density with respect to $F$.

**Lemma 2.** Suppose $\alpha > 0$ is given. Then there is a constant $C$ so that whenever $F$ is a closed subset of $G$ and $A(y, t)$ is any nonnegative measurable function on $G \times (0, \infty)$, then

$$\int_F \left( \int_{\Gamma_\alpha(x)} A(y, t) d\mu(y) dt \right) d\mu(x) \leq C \int_{\mathcal{R}_\alpha(F)} A(y, t) t^n d\mu(y) dt.$$

**Proof.** Fubini's theorem gives

$$\int_F \left( \int_{\Gamma_\alpha(x)} A(y, t) d\mu(y) dt \right) d\mu(x)$$

$$= \int_{G \times (0, \infty)} A(y, t) \left( \int_{\mathcal{F}} \chi_{B(y, \alpha t)}(x) d\mu(x) \right) d\mu(y) dt,$$
and so, for given \((y, t) \in R_\alpha(F)\), it will suffice to show that there is a constant \(C\) so that
\[
\int_F \chi_{B(y, \alpha t)}(x) d\mu(x) \leq Ct^n.
\]
In fact, let \((y, t) \in R_\alpha(F)\). Then
\[
\int_F \chi_{B(y, \alpha t)}(x) d\mu(x) \leq \int_G \chi_{B(y, \alpha t)}(x) d\mu(x)
= Ct^n,
\]
as desired. The proof is therefore complete.

**Lemma 3** [6]. Suppose \(\alpha > 0\) is given. Then there are constants \(C\) and \(\gamma, 0 < \gamma < 1\), sufficiently close to 1, so that whenever \(F\) is a closed subset of \(G\) and \(A(y, t)\) is a nonnegative measurable function on \(G \times (0, \infty)\), then
\[
\int_{R_\alpha(F^*)} A(y, t)t^n d\mu(y) dt \leq C \int_F \left( \int_{\Gamma(x)} A(y, t) d\mu(y) dt \right) d\mu(x),
\]
where \(F^*\) is the set of points of global \(\gamma\)-density with respect to \(F\).

**Lemma 4.** Let \(f\) be a nonnegative function defined on \(G\). Suppose
\[
\Lambda_t(f)(x) = \frac{1}{tn} \int_G \chi_{B(y, t)}(x) f(x) d\mu(x), \quad t > 0.
\]
Then there is a constant \(C\) such that
\[
\Lambda_{\alpha t}(f) \leq C \Lambda_t(M(f)).
\]
**Proof.** If \(f \geq 0\), then
\[
\Lambda_{\alpha t}(f) \leq C \Lambda_t(\Lambda_{\alpha t}(f)) \\
\leq C \Lambda_t(M(f)).
\]
since \(\Lambda_{\alpha t}(f) \leq CM(f)\). The proof is therefore complete.

Our main result is the following:
THEOREM 5. Let $0 < p < \infty$ and $\alpha > 1$. Then there is a constant $C$ such that

$$||S_\alpha(f)||_{L^p(d\mu)} \leq C||S(f)||_{L^p(d\mu)}.$$ 

with $S(f) = S_1(f)$.

Proof. Assume first that $0 < p < 2$. For each $\lambda > 0$, we define the open set $O$ by

$$O = ^cF = \{x \in G : S(f)(x) > \lambda\}.$$ 

Let $O^* = ^cF^*$. Then we take $F^* \subset F$ to be the set of points of global $\gamma$-density with respect to $F$. Apply Lemma 2 with $A(y,t) = |f(y,t)|^2t^{-n-1}$ (and $F^*$ in place of $F$), and we obtain

$$\int_{F^*} S_\alpha(f)(x)^2 d\mu(x) \leq C \int_{R_\alpha(F^*)} |f(y,t)|^2 \frac{d\mu(y) dt}{t}.$$ 

Next apply Lemma 3, again with $A(y,t) = |f(y,t)|^2 t^{-n-1}$, and we obtain

$$\int_{\mathcal{R}_\alpha(F^*)} |f(y,t)|^2 \frac{d\mu(y) dt}{t} \leq C \int_{F} \left( \int_{\Gamma(x)} |f(y,t)|^2 \frac{d\mu(y) dt}{t^{n+1}} \right) d\mu(x).$$ 

Then (3) and (4) imply that

$$\int_{F^*} S_\alpha(f)(x)^2 d\mu(x) \leq C \int_{F} S(f)(x)^2 d\mu(x).$$ 

Thus it follows from Lemma 1 and (5) that

$$\mu(\{x \in G : S_\alpha(f)(x) > \lambda\})$$

$$\leq \mu(O^*) + \frac{C}{\lambda^2} \int_{F} S(f)(x)^2 d\mu(x)$$

$$\leq C \left( \mu(O) + \frac{1}{\lambda^2} \int_{F} S(f)(x)^2 d\mu(x) \right)$$

$$= C \left( \mu(\{x \in G : S(f)(x) > \lambda\}) + \frac{1}{\lambda^2} \int_{F} S(f)(x)^2 d\mu(x) \right).$$
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Multiply both sides of (6) by \( \lambda^{p-1} \) and integrate, then we get that

\[
\|S_\alpha(f)\|_{L^p(d\mu)} \leq C\|S(f)\|_{L^p(d\mu)}
\]

for \( 0 < p < 2 \). Assume second that \( 2 \leq p < \infty \). Then observe that

\[
\|S_\alpha(f)\|_{L^p(d\mu)}^2 = \sup_{\psi} \int_G S_\alpha(f)(x)^2\psi(x)d\mu(x),
\]

where the supremum is taken over all \( \psi \) which belong to \( L^r(d\mu) \) with \( r \) dual to \( 2/p \), and \( \|\psi\|_{L^r(d\mu)} \leq 1 \). Then it follows from Lemma 4 that

\[
(7) \quad \int_G S_\alpha(f)(x)^2\psi(x)d\mu(x)
\]

\[
= \int_{G \times (0,\infty)} |f(y,t)|^2 \Lambda_\alpha(\psi)(y)\frac{d\mu(y)dt}{t}
\]

\[
\leq C \int_{G \times (0,\infty)} |f(y,t)|^2 \Lambda_1(M(\psi))(y)\frac{d\mu(y)dt}{t}
\]

\[
= C \int_G S(f)(x)^2M(\psi)(x)d\mu(x)
\]

\[
\leq C\|S(f)\|_{L^p(d\mu)}^2\|M(\psi)\|_{L^r(d\mu)}
\]

\[
\leq C\|S(f)\|_{L^p(d\mu)}^2\|\psi\|_{L^r(d\mu)} (\text{by } r > 1)
\]

\[
\leq C\|S(f)\|_{L^p(d\mu)}^2.
\]

Taking the supremum over all \( \psi \) in (7). Then

\[
\|S_\alpha(f)\|_{L^p(d\mu)} \leq C\|S(f)\|_{L^p(d\mu)}
\]

with \( 2 \leq p < \infty \). The proof is therefore complete.

References

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