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ON THE SUBFACTORS RELATED TO GROUP ACTIONS

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Introduction

After V. Jones's work ([4]) on index for subfactors, the classsification of the subfactors of a type II_1 factor has been one of the central subjects in the theory of operator algebras.

In the theory of subfactors, principal graphs and, more generally, paragroups of inclusions are most important invariants for finite index subfactors with trivial relative commutants (see [2], [9], [13]). Unfortunately, direct computation of the principal graphs is quite difficult even for the subfactors constructed explicitly. For this reason, several authors have tried to understand a more managable case about subfactors associated to crossed product construction by group actions, more generally Hopf algebra actions ([3], [9], [14]). In particular, for crossed products by a finite group, it is well-known that finite group theory and the representation theory of groups determine the principal graphs ([6], [7]).

D. Bisch and U. Haagerup have recently investigated in [1] a class of subfactors $P^H \subset P \times K$, where the finite groups H and K act outerly on a hyperfinite type II_1 factor P. By using bimodule techniques, they studied the properties like irreducibility, finite depth (see [2] for the definitions), amenability, and strong amenability (in the sense of S. Popa in [13]) of the inclusion. Also they were succesful to provide various examples of subfactors. In spite of those examples of subfactors, it is still difficult to sketch the explicit algorithm of the paragroups for $P^H \subset P \times K$ as in the case of crossed products construction, for example $P \subset P \times G$ by a finite group G outer action. In this paper, we are going to compute the basic construction of the inclusion

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$P^H \subset P \times K$, when P is an arbitrary type II_1 factor. As a consequence, this will provide a better understanding of the principal graphs for the inclusion.

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2.Preliminaries

2.1 Group actions and subfactors

Let P be a type II_1 factor with a unique normalized trace τ. Let H and K be finite groups acting outerly on P via

$$\alpha : K \longrightarrow Aut(P) \quad \text{and} \quad \beta : H \longrightarrow Aut(P)$$

respectively. In this section we quote some preliminary materials and fix notations which will be needed in the sequel.

1 Identities in all algebras will be denoted by I. The complex numbers are denoted by C. And $C[G]$ denotes the complex group algebra of a finite group G.

2 For $s \in K$, $\alpha_s(x) = sxs^{-1}$, $(x \in P)$, where s is viewed as a unitary element implementing the automorphism α_s . Then $P \times_\alpha K = \{\sum_{s\in K} x_s s \mid x_s \in P\}$.

3 Similarily, $\beta_h(x) = hxh^{-1}$ $(x \in P)$, for $h \in H$. Then P^H denotes the fixed point algebra of P under the action β.

4 The vector space K° of linear functionals on $C[K]$ is a Hopf $*-$ algebra dual to K. Then there is a dual action $\hat{\alpha}$ of K° on $P \times_\alpha K$, defined by

$$\hat{\alpha}_k(xs) = xk(s)s, \text{ for } s \in K, \text{ and } x \in P,$$

when k is an element of K°. Then the dual crossed product is given by

$$(P \times_\alpha K) \times_{\hat{\alpha}} K^\circ \stackrel{\text{def}}{=} P_1 = \{\sum_{t\in K} y_t p_t \mid y_t \in P \times_\alpha K\},$$

where p_t denotes the minimal projection in K°. Hence

$$P_1 = \{ \sum_{s,t \in K} x_{s,t} s p_t \mid x_{s,t} \in P \},$$

and $\hat{\alpha}_k(p_s) = k p_s k^{-1} = p_{ks}$ for $k \in K^\circ$.

Since the involved actions are outer, it is well-known that P^H and $P \times_\alpha K$ are type II_1 factors with $(P^H)' \bigcap P = CI$, $P' \bigcap (P \times_\alpha K) = CI$.

Let G be the group generated by H and K in the outer automorphisms $Out(P) = Aut(P)/Int(P)$, where $Int(P)$ denotes the set of inner automorphisms of P. It is shown that some properties of the inclusion can be expressed in terms of properties of the group G, by using the bimodule techniques ([1]). In this article, our aim is to study the inclusion

$$P^H \subset P \times_\alpha K,$$

based on the representation theory of finite groups. Since $P^H \subset P \subset P \times_\alpha K$, the Jones index ([4]) of the inclusion is

$$[P \times_\alpha K : P^H] = [P \times_\alpha K : P][P : P^H] = |K||H|,$$

where $|\cdot|$ denotes the order of a group.

2.2 The algebra P_1

At first we describe the algebra P_1, with the help of Y. Nakagami and M. Takesaki's duality theorem in [8] and some results in [10]. We denote by $\mathcal{L}^2(K)$ the Hilbert space, whose inner product $< a, b >= \varphi(b^* a)$ with the Haar trace φ of K ($a, b \in K$). Also we denote by $End(\mathcal{L}^2(K))$ the C^*- algebra of linear endomorphisms of $\mathcal{L}^2(K)$.

LEMMA 2.1 [10]. *Let $Q = \{K, K^\circ\}' \bigcap P_1$, where $'$ denotes the commutants. Then the following hold.*

1 *The subalgebra $\{K, K^\circ\}''$ of P_1, generated by K and K°, is isomorphic to $End(\mathcal{L}^2(K))$.*
2 *There is a $*-$ isomorphism from P onto Q, given by $x \mapsto \sum_{s \in K} \alpha_s(x) p_s$, for $x \in P$.*

Proof. Note that K° acts on K via $k \cdot s = k(s)s$, for $s \in K$ and $k \in K^\circ$. Thus the crossed product algebra is $K \times K^\circ = \{K, K^\circ\}''$. Then there is a $*-$ isomorphism from $K \times K^\circ$ onto $End(\mathcal{L}^2(K))$. This gives the first assertion.

For the second assertion, choose $q = \sum_{s,t} x_{s,t} s p_t \in Q$, with $x_{s,t} \in P$. Then q commutes with K if and only if $x_{tr^{-1},rs} = \alpha_r(x_{r^{-1}t,s})$ for $r, t, s \in K$. Also q commutes with K° if and only if $x_{s,s^{-1}t} = 0$ for $s(\neq e), t \in K$. Therefore $Q = \{\sum_{s\in K} \alpha_s(x)p_s | x \in P\}$. Then the map given by $x \mapsto \sum_{s\in K} \alpha_s(x)p_s$ determines a $*-$ isomorphism from P onto Q, for $x \in P$.

The following theorem is essentially the Duality theorem of Y. Nakagami and M. Takesaki ([8]).

THEOREM(DUALITY THEOREM). $P_1 \cong Q \otimes End(\mathcal{L}^2(K))$.

Proof. It follows from tha fact 1 of Lemma 2.1 that $P_1 \cong P \otimes End(\mathcal{L}^2(K))$. Therefore, by the fact 2 of Lemma 2.1, $P_1 \cong Q \otimes End(\mathcal{L}^2(K))$.

2.3 An action σ of H on P_1

To estabilish a crossed product of P_1 by H, we define an action σ of H on P_1 as follows;

DEFINITION 2.1. For $h \in H$, the action $\sigma_h : H \longrightarrow Aut(P_1)$ is defined by

1 $\sigma_h(s) = s$, for $s \in K$,
2 $\sigma_h(p_s) = p_s$, for $s \in K$,
3 $\sigma_h(\sum_{s\in K} \alpha_s(x)p_s) = \sum_{s\in K} \alpha_s(\beta_h(x))p_s$, for $x \in P$.

From now on, $\delta_{a,b}$ denotes the Kronecker's delta as usual.

PROPOSITION 2.1. *If $x \in P$, then we have*
$\sigma_h(x) = \sum_{s\in K}(\alpha_s\beta_h\alpha_s^{-1})(x)p_s$. *Also $\sigma_h : H \longrightarrow Aut(P_1)$ is outer if $h \neq e$.*

Proof. Let $x \in P$. Then

$$\begin{aligned}\sigma_h(x) &= \sigma_h(\sum_{s\in K} xp_s) \\ &= \sigma_h[\sum_{s\in K} \alpha_s(\alpha_s^{-1}(x))p_s] \\ &= \sum_{s\in K} \alpha_s[\beta_h(\alpha_s^{-1}(x))]p_s \\ &= \sum_{s\in K} (\alpha_s\beta_h\alpha_s^{-1})(x)p_s\end{aligned}$$

by the property 3 in Definition 2.1. The outerness of σ now comes from the outerness of α and β.

Therefore, the crossed product algebra

$$P_1 \times_\sigma H = \{ \sum_{s,t\in K, h\in H} x_{s,t,h} s p_t h | x_{s,t,h} \in P\}$$

is a type II_1 factor. Note that the normalized trace Tr on $P_1 \times_\sigma H$ is given by

$$Tr(xsp_th) = \frac{\tau(x)}{|K|}\delta_{s,e}\delta_{h,e}, \text{ for } x \in P.$$

where τ denotes the normalized trace on P.

3.The Jones tower for $P^H \subset P \times_\alpha K$

3.1 The Jones basic construction

In this section, we are going to show that $P_1 \times_\sigma H$ is the desired basic construction for $P^H \subset P \times_\alpha K$ under a proper assumption. In order to compute the basic construction for $P^H \subset P \times_\alpha K$, it is enough to find the Jones projection for the inclusion. Note that the corresponding Jones projection lies in the relative commutant algebra of P^H in the basic construction. We need the following observations for the element's form of the Jones projection.

LEMMA 3.1. *Let $s, t \in K$, $h \in H$, and θ be an arbitrary outer action of K on P. If $x_{s,t,h} \in P$ satisfies $x_{s,t,h}\theta(y) = yx_{s,t,h}$ for all $y \in P^H$, then there is a function $u_{s,t,h} : K \times K \times H \longrightarrow \mathcal{U}(P)\bigcup\{0\}$ such that*

$$\theta(y) = u^*_{s,t,h} y u_{s,t,h}, \text{ for all } y \in P^H,$$

whenever $u_{s,t,h} \neq 0$. Here $\mathcal{U}(P)$ denotes the set of unitaries in P.

Proof. For $s, t \in K$ and $h \in H$, let $x_{s,t,h}\theta(y) = yx_{s,t,h}$ for $x_{s,t,h} \in P$, $y \in P^H$. Then $\theta(y)x^*_{s,t,h} = x^*_{s,t,h}y$, by taking the adjoint of both sides. Since

$$x^*_{s,t,h} \cdot x_{s,t,h}\theta(y) = x^*_{s,t,h}y \cdot x_{s,t,h} = \theta(y)x^*_{s,t,h}x_{s,t,h},$$

we see that

$$x^*_{s,t,h}x_{s,t,h} \in [\theta(P^H)]' \bigcap P \text{ or } \theta^{-1}(x^*_{s,t,h}x_{s,t,h}) \in (P^H)' \bigcap P = CI$$

. Hence, there is a scalar $\lambda \in C$ and a unitary $u_{s,t,h} \in \mathcal{U}(P)\bigcup\{0\}$ such that $x_{s,t,h} = \lambda u_{s,t,h}$, if $u_{s,t,h} \neq 0$.

Thus, $x_{s,t,h}\theta(y) = \lambda u_{s,t,h}\theta(y)$ and $yx_{s,t,h} = y\lambda u_{s,t,h}$. It now follows that $u_{s,t,h}\theta(y) = yu_{s,t,h}$, or $\theta(y) = u^*_{s,t,h}yu_{s,t,h}$.

Due to Lemma 3.1, we now give an explicit proof of the following fact, described in [1] without proof. $\mathcal{T}$ will denote the torus.

PROPOSITION 3.1. *There is a 2-cocycle $\omega : (H\cap K)\times(H\cap K) \longrightarrow \mathcal{T}$ such that $(P^H)' \bigcap(P \times_\alpha K) \cong C_\omega[H \bigcap K]$, the complex group algebra twisted by ω*

Proof. Let $\sum_{s\in K} x_s s \in (P^H)' \bigcap(P \times_\alpha K)$, with $x_s \in P$. If $y \in P^H$, then

$$\sum_{s\in K} x_s s \cdot y = y \cdot \sum_{s\in K} x_s s, \text{ or } \sum_{s\in K} x_s\alpha_s(y)s = \sum_{s\in K} yx_s s.$$

So we have $x_s\alpha_s(y) = yx_s$ for all $s \in K$. By Lemma 3.1, there exists a unitary element $u_s \in P$ such that $\alpha_s(y) = u^*_s y u_s$, if $u_s \neq 0$. Thus $x_s u^*_s y = yx_s u^*_s$, and so $x_s u^*_s \in (P^H)' \bigcap P \cong C$, if $u_s \neq 0$. Therefore

$x_s = \lambda_s u_s$ for some $\lambda_s \in C$, if $u_s \neq 0$. This implies that $u_s\alpha_s(y) = y$ for $y \in P^H$, or $adu_s\alpha_s \in \mathcal{G}(P^H, P) = \beta(H)$. Here $\mathcal{G}(P^H, P$ denotes the Galois group of the inclusion $P^H \subset P$ (see [5]). Hence there is a $h \in H$ such that $\beta_h = adu_s\alpha_s$, i.e., $\alpha_s = \beta_h$ in $Out(P)$, for $s \in K$ and $h \in H$. In other words,

$$(P^H)' \bigcap (P \times_\alpha K) = \{\sum_{s \in K} \lambda_s u_s s | s \in H \cap K \text{ in } Out(P)\}.$$

On the other hand, if $y \in P^H$, then

$$u_s u_t u_{st}^* y u_{st} u_t^* u_s^* = (\alpha_s^{-1}\alpha_t^{-1}\alpha_{st})(y) = y$$

This implies that $u_s u_t u_{st}^* \in (P^H)' \bigcap P = CI$. Therefore there is a 2-cocycle $\omega : (H \cap K) \times (H \cap K) \longrightarrow \mathcal{T}$, given by $u_s u_t = \omega(s,t) u_{st}$, such that

$$(P^H)' \bigcap (P \times_\alpha K) \cong C_\omega[H \cap K].$$

This completes the proof.

The following corollary comes immediately.

COROLLARY 3.1. *$P^H \subset P \times_\alpha K$ is irreducible if and only if $H \bigcap K = \{e\}$ in $Out(P)$, where e denotes the neutral element of the group G.*

From now on, we assume that $H \bigcap K = \{e\}$ in $Out(P)$. Under this assumption, the unitaries $u_{s,t,h}$ in Lemma 3.1 also determine the form of elements in $(P^H)' \bigcap (P_1 \times_\sigma H)$.

LEMMA 3.2. *If $H \bigcap K = \{e\}$ in $Out(P)$, then every element of the algebra $(P^H)' \bigcap (P_1 \times_\sigma H)$ has the form*

$$\sum_{s,t \in K, h \in H} \lambda_{s,t,h} u_{s,t,h} s p_t h (\lambda_{s,t,h} \in C).$$

Proof. Let $y \in P^H$ commute with $\sum_{s,t \in K, h \in H} x_{s,t,h} s p_t h \in P_1 \times_\sigma H$, with $x_{s,t,h} \in P$. Then we have

$$y \sum_{s,t \in K, h \in H} x_{s,t,h} s p_t h = \sum_{s,t \in K, h \in H} x_{s,t,h} s p_t h \cdot y$$

implies that

$$\begin{aligned}
\sum_{s,t\in K,h\in H} y\cdot x_{s,t,h}sp_th\cdot h^{-1} &= \sum_{s,t\in K,h\in H} x_{s,t,h}sp_thy\cdot h^{-1}\\
&= \sum_{s,t\in K,h\in H} x_{s,t,h}sp_t\sigma_h(y)\\
&= \sum_{s,t\in K,h\in H} x_{s,t,h}s\sigma_h(y)p_t\\
&= \sum_{s,t\in K,h\in H} x_{s,t,h}s[\sum_{g\in K}(\alpha_g\beta_h\alpha_g^{-1})(y)p_g]p_t\\
&= \delta_{g,t}\sum_{s,t\in K,h\in H} x_{s,t,h}s(\alpha_t\beta_h\alpha_t^{-1})(y)p_t\\
&= \delta_{g,t}\sum_{s,t\in K,h\in H} x_{s,t,h}\alpha_s((\alpha_t\beta_h\alpha_t^{-1})(y))sp_t\\
&= \delta_{g,t}\sum_{s,t\in K,h\in H} x_{s,t,h}(\alpha_{st}\beta_h\alpha_t^{-1})(y)sp_t,
\end{aligned}$$

by the property 3 of Definition 2.1 and the fact that $p_t\in P'$. Thus, for all $s,t\in K$ and $h\in H$, we have $yx_{s,t,h}=x_{s,t,h}(\alpha_{st}\beta_h\alpha_t^{-1})(y)$. Then, by Lemma 3.1, there are unitaries $u_{s,t,h}\in\mathcal{U}(P)\bigcup\{0\}$ such that $(\alpha_{st}\beta_h\alpha_t^{-1})(y)=u^*_{s,t,h}yu_{s,t,h}$, if $u_{s,t,h}\neq 0$. Therefore

$$x_{s,t,h}u^*_{s,t,h}yu_{s,t,h}=yx_{s,t,h},\text{ or } x_{s,t,h}u^*_{s,t,h}y=yx_{s,t,h}u^*_{s,t,h}.$$

i.e. $x_{s,t,h}u^*_{s,t,h}\in(P^H)'\bigcap P=\mathcal{C}I$, and so we see that $x_{s,t,h}=\lambda_{s,t,h}u_{s,t,h}$ for some scalar $\lambda_{s,t,h}$. This completes the proof.

Note that the trace preserving conditional expectation $E:P\times_\alpha K\longrightarrow P^H$ is given by

$$E(\sum_{s\in K}x_ss)=E^P_{P^H}E^{P\times K}_P(\sum_{s\in K}x_ss)=E^P_{P^H}(x_e)=\frac{1}{|H|}\sum_{h\in H}\beta_h(x_e),$$

where e denotes the neutral element of a group G. Also, the trace preserving conditional expectation $F:P_1\times_\sigma H\longrightarrow P\times_\alpha K$ is given by

$$F(\sum_{s,t\in K,h\in H}x_{s,t,h}sp_th)=\frac{1}{|K|}\sum_{s\in K}x_{s,e,e}s,$$

in a similar way. We now can determine the Jones projection due to Lemma 3.2, i.e., it has of the form $\sum_{s,t\in K,h\in H}\lambda_{s,t,h}u_{s,t,h}sp_t h$, with $\lambda_{s,t,h}\in C$.

PROPOSITION 3.2. *Let $H\bigcap K=\{e\}$ in $Out(P)$ Then*

$$q=\frac{1}{|H|}\sum_{h\in H}p_e h\in P_1\times_\sigma H$$

is the Jones projection for $P^H\subset P\times_\alpha K$.

Proof. Since p_t commutes with $h\in H$, the straightforward computation gives that q is a projection in $P_1\times_\sigma H$. Also, q commutes with P^H. Indeed, for $y\in P^H$,

$$y\cdot q=\frac{1}{|H|}\sum_{h\in H}yp_e h=\frac{1}{|H|}\sum_{h\in H}yhp_e.$$

But $hyp_e=\sigma_h(y)hp_e=\sum_{t\in K}(\alpha_t\beta_h\alpha_t^{-1})(y)p_t\cdot hp_e=\sum_{t\in K}(\alpha_t\beta_h\alpha_t^{-1})$ $(y)hp_tp_e=\delta_{t,e}\beta_h(y)hp_e=yhp_e$, for $y\in P^H$. Therefore,

$$y\cdot q=\frac{1}{|H|}\sum_{h\in H}yhp_e=\frac{1}{|H|}\sum_{h\in H}hyp_e=\frac{1}{|H|}\sum_{h\in H}hp_e y=q\cdot y,$$

because $y\in P$ and $p_e\in P'$.

It now suffices to show $qzq=E(z)q$ for $z\in P\times_\alpha K$. Let $x_s s\in$

$P \times_\alpha K$, with $x_s \in P$. Then

$$
\begin{aligned}
q(x_s s)q &= \frac{1}{|H|^2} \sum_{h,g\in H} p_e h \cdot x_s s \cdot p_e g \\
&= \frac{1}{|H|^2} \sum_{h,g\in H} h p_e \cdot x_s s \cdot p_e g \\
&= \frac{1}{|H|^2} \sum_{h,g\in H} h x_s \cdot p_e s p_e \cdot g \\
&= \frac{1}{|H|^2} \sum_{h,g\in H} h x_s \cdot s p_{s^{-1}} p_e g \\
&= \delta_{s,e} \frac{1}{|H|^2} \sum_{h,g\in H} \sigma_h(x_e) h g p_e \\
&= \delta_{s,e} \frac{1}{|H|^2} \sum_{h,g\in H} [\sum_{t\in K} (\alpha_t \beta_h \alpha_t^{-1})(x_e) p_t] h g p_e \\
&= \delta_{s,e} \frac{1}{|H|^2} \sum_{h,g\in H} (\sum_{t\in K} (\alpha_t \beta_h \alpha_t^{-1})(x_e) h g p_t p_e \\
&= \delta_{t,e} \delta_{s,e} \frac{1}{|H|^2} \sum_{h,g\in H} \beta_h(x_e) h g p_e \\
&= \delta_{t,e} \delta_{s,e} [\frac{1}{|H|} \sum_{h\in H} \beta_h(x_e)] \cdot [\frac{1}{|H|} \sum_{h,g\in H} p_e h g] \\
&= E(x_s s) q.
\end{aligned}
$$

This completes the proof.

Note that

$$F(q) = F(\frac{1}{|H|}\sum_{h\in H} p_e h) = \frac{1}{|H|}F(\sum_{h\in H} p_e h) = \frac{1}{|H|}\frac{1}{|K|}I$$
$$= [P \times_\alpha K : P^H]^{-1} I,$$

and

$$Tr(q) = Tr(\frac{1}{|H|}\sum_{h\in H} p_e h) = \frac{1}{|H|}Tr(\sum_{h\in H} p_e h) = \frac{1}{|H|}\frac{1}{|K|}$$
$$= [P \times_\alpha K : P^H]^{-1}.$$

Now we are in a position to conclude the basic construction for $P^H \subset P \times_\alpha K$.

THEOREM 3 1. *Let $H \bigcap K = \{e\}$ in $Out(P)$. The basic construction for the inclusion $P^H \subset P \times_\alpha K$ is isomorphic to $P_1 \times_\sigma H$*

Proof.. By Proposition 3.2, the projection $q \in P_1 \times_\sigma H$ satisfies
1 $q \in (P^H)'$,
2 $F(q) = [P \times_\alpha K : P^H]^{-1} I$,
where F denotes the trace preserving conditional expectation from $P_1 \times_\sigma H$ onto $P \times_\alpha K$. Then it follows from [4] (or [11]) that $P_1 \times_\sigma H$ is isomorphic to the basic construction for $P^H \subset P \times_\alpha K$.

3.2 The Jones tower

Let $M_{-1} = P^H$, $M_0 = P \times_\alpha K$ and $M_1 = \langle M, e_{-1}\rangle = P_1 \times_\sigma H$, where e_{-1} denotes the corresponding Jones projection for the inclusion $M_{-1} \subset M_0$. Note that η_s $(s \in K)$, given by

$$\eta_s(xtp_r) = xtp_{rs^{-1}}, \text{ for } x \in P, t, r \in K,$$

determines an outer action $\eta : K \longrightarrow Aut(P_1)$.

LEMMA 3.3. *$P \times_\alpha K = M_0 = (P_1)^K$, the fixed point algebra under the action η of K on P_1.*

Proof. Since η fixes P and $s \in K$, it is clear that $P \times_\alpha K \subset (P_1)^K \subset P_1$. Thus we have

$$[P_1 : (P_1)^K] = [P_1 : P \times_\alpha K] = |K|,$$

and so $[(P_1)^K : P \times_\alpha K] = 1$. Hence, the result follows from the properties of Jones index (see [4]).

With the help of Lemma 3.3, we conclude that there are outer actions

$$\sigma : H \longrightarrow Aut(P_1), \eta K \longrightarrow Aut(P_1),$$

such that $M_0 = (P_1)^K \subset P_1 \times_\sigma H = M_1$. This situation is exactly like $M_{-1} \subset M_0$, when M_{-1} is replaced by M_0, M_0 replaced by M_1, P replaced by P_1, K replaced by H, and H replaced by K. The next result follows immediately after repeating the same process as in Section 3.1. Also note that there is a dual action $\hat{\sigma}$ of the dual Hopf $*-$ algebra H° on $P_1 \times_\sigma H$.

PROPOSITION 3.3. *Let $P_2 = (P_1 \times_\sigma H) \times_\sigma H^\circ$. Then, $\rho_s (s \in K)$, given by*

1 $\rho_s(h) = h$, *for* $h \in H$,
2 $\rho_s(p_h) = p_h$ *for* $h \in H$,
3 $\rho_s(x) = \sum_{h \in H} (\sigma_h \eta_s \sigma_h^{-1})(x) p_h$, *for* $x \in P_1$,
4 $\rho_s(\sum_{h \in H} \sigma_h(x) p_h) = \sum_{h \in H} \sigma_h(\eta_s(x)) p_h$, *for* $x \in P_1$,

determines an outer action ρ of K on P_2 such that $M_2 = P_2 \times_\rho K$.

For the Jones tower, we use $P \times K$ instead of $P \times_\alpha K$ by dropping the involved actions in our notations, since no confusion is possible.

COROLLARY 3.2. *Let*

$$M_{-1} = P^H,\ M_0 = P \times K \text{ and } M_n = \langle M_{n-1}, e_{n-1} \rangle,$$

where e_n denotes the corresponding Jones projection for $M_{n-2} \subset M_{n-1}$, $(n \geq 0)$. Then

$$M_{2n} \cong P_{2n} \times K,$$

$$M_{2n+1} \cong P_{2n+1} \times H,$$

where $P_{2n} = (P_{2n-1} \times K) \times K^\circ$ and $P_{2n+1} = (P_{2n} \times H) \times H^\circ$ $(n \geq 0)$. Here the involved actions on P_n are as described in Theorem 3.1.

Proof. The result follows from Theorem 3.1 and 3.2, inductively.

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