IRREDUCIBLE MODULES FOR
SOME METACYCLIC GROUPS

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The aim of this note is to give an explicit description of all isomorphism types of irreducible modules over a finite field for a metacyclic group presented by \( \langle x, y \mid x^m = 1, y^n = 1, y^{-1}xy = x^r \rangle \) where \( q \) is a prime and \( r \) is a \( q \)-th roots of 1 modulo \( m \). The main results of this note generalize the investigation by Barlotti [1] for metacyclic groups of order \( pq \) (\( p, q \) . primes).

1. Background Results

We first set up some notation which will be kept throughout this note. Let \( F \) be a finite field and let \( a(n) \) denote the multiplicative order of \( |F| \) modulo \( n \) for every positive integer \( n \). Let \( G(m, n) \) be a metacyclic group defined by

\[
G(m, n) = \langle x, y \mid x^m = 1, y^n = 1, y^{-1}xy = x^r \rangle
\]

where \( r \) is a primitive \( n \)-th root of 1 modulo \( m \); note that all possible such \( r \) give the same group for the fixed integers \( m \) and \( n \). When \( n \) is a prime, for each positive divisor \( d \) of \( m \) the group defined by

\[
\langle x, y \mid x^d = 1, y^n = 1, y^{-1}xy = x^r \rangle
\]

is \( G(d, n) \) provided that \( d \) does not divide \( r - 1 \), while the group is abelian if \( d \) divides \( r - 1 \).

Most of notation and terminology which are not defined in this note are standard, or can be found in [3] or [2].

We continue with some important construction of faithful irreducible modules for the group \( G(m, n) \).

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CONSTRUCTION 1.1. Let \( m \) be a positive integer not divisible by the characteristic of \( \mathbb{F} \) and let \( n \) be a divisor of \( a(m) \). Let \( K \) be the field with \( |\mathbb{F}|^{a(m)} \) elements, let \( u \) be an element of multiplicative order \( m \) in \( K \), and write \( V \) for the \( K \) viewed as a vector space over \( \mathbb{F} \).

(a) There is an action of \( G(m,n) \) on \( V \) such that for every \( v \) in \( V \),

\[ vx = vu \quad \text{and} \quad vy = v^{a(m)/n}. \]

(b) Under the action in (a), \( V \) is a faithful irreducible module for \( G(m,n) \) over \( \mathbb{F} \); denote the module by \( V(u) \).

(c) \( \text{End}_{\mathbb{F}G(m,n)} V(u) \) is the field with \( |\mathbb{F}|^{a(m)/n} \) elements; in particular, if \( n = a(m) \) then \( V(u) \) is an absolutely irreducible faithful module for \( G(m,n) \) over \( \mathbb{F} \).

Note that this construction is well known for \( d = 1 \) from the representation theory of cyclic groups, while the proof for the general case can be found in [1]. It is also well known that every faithful irreducible module for a finite cyclic group (say, \( G(m,1) \) here) is realized as such a module described in this construction, and \( V(u) \) and \( V(v) \) are isomorphic if and only if \( u \) and \( v \) are roots of the same irreducible factor of \( x^m - 1 \) in \( \mathbb{F}[x] \).

Let \( A \) be a finite abelian group and \( V \) an irreducible \( \mathbb{F}A \)-module. The factor group of \( A \) by the kernel \( \{ g \in A : vg = v \text{ for all } v \in V \} \) of \( V \) is cyclic. Conversely, every subgroup of \( A \) with cyclic quotient becomes the kernel of a certain irreducible \( \mathbb{F}A \)-module, provided that the characteristic of \( \mathbb{F} \) does not divide the order of the cyclic quotient. This leads to a complete description of the irreducible modules for a finite abelian group over a finite field.

Suppose that the abelian group \( A \) is metacyclic. Then \( A \) is a direct product of two finite cyclic groups \( C_m \) and \( C_n \) for some nonnegative integers \( m \) and \( n \) such that \( n \) divides \( m \). For any positive divisor \( d \) of \( m \) we define \( \#(d) \) to be the number of all cyclic quotients of order \( d \) of \( A \). If the characteristic of \( \mathbb{F} \) does not divide \( m \), there exists precisely \( \sum_{d|m} \#(d) \cdot \phi(d)/a(d) \) pairwise nonisomorphic irreducible modules for \( A \) over \( \mathbb{F} \).

2. Main Results
Let \( p \) be the characteristic of \( F \), let \( q \) be a fixed prime, let \( m \) be a fixed positive integer and let \( d \) be a positive divisor of \( m \). Let \( G \) be a finite group whose factor group by the largest normal \( p \)-subgroup \( O_p(G) \) is isomorphic to \( G(m, q) \). Since \( O_p(G) \) is contained in the kernels of all irreducible \( FG \)-modules, there is a natural one-to-one correspondence between the irreducible \( FG \)-modules and the irreducible \( FG(m, q) \)-modules.

We now consider faithful irreducible modules for \( G(m, q) \) whose order is not divisible by \( p \). The cyclic normal subgroup generated by \( x \) in \( G(m, q) \) is denoted by \( M \).

**Theorem 2.1.** If \( q \) divides \( a(m) \), every faithful irreducible module for \( G(m, q) \) over \( F \) is isomorphic to an \( FG(m, q) \)-module described in Construction 1.1. So there exist precisely \( \phi(m)/a(m) \) isomorphism types of faithful irreducible modules for \( G(m, q) \) over \( F \).

**Proof.** Let \( V_1, \ldots, V_n \) be pairwise nonisomorphic faithful irreducible modules for \( M \) over \( F \), where \( n = \phi(m)/a(m) \). Let \( W_1, \ldots, W_n \) be the faithful irreducible modules for \( G(m, q) \) over \( F \), as described in Construction 1.1, such that \( (W_i)_M \cong V_i \) for all \( i = 1, \ldots, n \). Then \( FM = V_0 \oplus V_1 \oplus \cdots \oplus V_n \) for some \( FM \)-module \( V_0 \). It follows that \( FG(m, q) \cong V_0^{G(m, q)} \oplus V_1^{G(m, q)} \oplus \cdots \oplus V_n^{G(m, q)} \). For each \( i = 1, \ldots, n \), the multiplicity of \( W_i \) as a composition factor in the head of \( V_i^{G(m, q)} \) is \( (\dim \text{End}_{FM} V_i)/(\dim \text{End}_{FG(m, q)} W_i) = a(m)/(a(m)/q) = q \) by Construction 1.1 (c) and Theorem 4.13 in [2]. Therefore, \( V_i^{G(m, q)} \) is isomorphic to the direct sum of \( q \) copies of \( W_i \).

Let \( W \) be an irreducible \( FG(m, q) \)-module which is not isomorphic to \( W_i \) for all \( i = 1, \ldots, n \). Then \( W \) is a homomorphic image of \( V_i^{G(m, q)} \) for some irreducible submodule \( V \) of \( V_0 \), and hence \( V \) is isomorphic to a submodule of \( W_M \). It follows that \( \text{Ker} W \cong \text{Ker} V^{G(m, q)} = \text{Core}_{G(m, q)} \text{Ker} V = \text{Ker} V \neq 1 \), which implies \( W \) is not faithful. Consequently, every faithful irreducible module for \( G(m, q) \) over \( F \) is isomorphic to one of the \( W_i \). □

**Lemma 2.2.** Let \( V \) a faithful irreducible module for \( M \) over \( F \). If \( q \) does not divide \( a(m) \), then \( V \) is not isomorphic to \( V \otimes y \).

**Proof.** There are precisely \( \phi(m)/a(m) \) isomorphism types of faithful irreducible modules for \( M \) over \( F \), which are transitively permuted by
Aut $M$. It follows that the stabilizer in Aut $M$ of the isomorphism type of $V$ is a subgroup of index $\phi(m)/a(m)$ in Aut $M$ (equivalently, of order $a(m)$).

The statement $V \cong_{FM} V \otimes y$ says that the element which maps $x$ to $x^r$ (of order $q$) in Aut $M$ lies in this subgroup of order $a(m)$. It follows that $V \cong_{FM} V \otimes y$ implies $y \mid a(m)$. □

**Theorem 2.3.** If $q$ does not divide $a(m)$, then
   (a) every $FG(m,q)$-module induced from a faithful irreducible module for $M$ over $F$ is faithful and irreducible;
   (b) every faithful irreducible module for $G(m,q)$ over $F$ is induced from a faithful irreducible module for $M$ over $F$.

**Proof.** (a) Let $V$ be a faithful irreducible module for $M$ over $F$. Then $V^{G(m,q)}$ is faithful, since the kernel of $V^{G(m,q)}$ is the core of the kernel of $V$ in $G(m,q)$. By Lemma 2.2 and Theorem 9.6 b) in [3], $V^{G(m,q)}$ is irreducible.

(b) Let $V_1, \ldots, V_n$ be the $\phi(m)/a(m)$ pairwise nonisomorphic faithful irreducible modules for $M$ over $F$. Suppose $FM = V_0 \oplus V_1 \oplus \cdots \oplus V_n$. Then $FG(m,q) \cong V_0^{G(m,q)} \oplus V_1^{G(m,q)} \oplus \cdots \oplus V_n^{G(m,q)}$. No irreducible constituent of $V_0$ is faithful, so every faithful irreducible module for $G(m,q)$ over $F$ is isomorphic to one of the $V_i^{G(m,q)}$. □

**Corollary 2.4.** Assume that the characteristic of $F$ does not divide $d$. There exist precisely $\phi(d)/[a(d),q]$ isomorphism types of faithful irreducible modules for $G(d,q)$ over $F$, where $[a(d),q]$ is the least common multiple of $a(d)$ and $q$.

**Proof.** If $q$ divides $a(d)$, then from Theorem 2.1, there exist precisely $\phi(d)/a(d)$ isomorphism types of faithful irreducible modules for $G(d,q)$ over $F$.

If $q$ does not divide $a(d)$, then $V_i \cong V_i \times y^j$ for all $j = 0, \ldots, q - 1$, by Lemma 2.2. Since $V_i^{G(m,q)} \cong V_j^{G(m,q)}$ if and only if $V_i \cong V_j \otimes y^k$ for some $k = 0, \ldots, q - 1$, the multiplicity of $V_i$ as a composition factor in $V_1^{G(m,q)} \oplus \cdots \oplus V_n^{G(m,q)}$ is $q$ for all $i = 1, \ldots, n$. Hence there are exactly $\phi(d)/a(d)q$ isomorphism types of faithful irreducible modules for $G(d,q)$ over $F$. □

Let $d_0$ be the greatest common divisor $m$ and $r - 1$, and let $\Delta$ be
the set of all positive divisors of $m$ which do not divide $r - 1$. Then we have

**Theorem 2.5.** Let $G$ be a finite group whose factor group by the largest normal $p$-subgroup is isomorphic to $G(m, q)$. There is a one-to-one correspondence between the set of isomorphism types of all irreducible $\mathbb{F}G$-modules and the union of the following two sets: (i) the set of isomorphism types of all faithful irreducible $\mathbb{F}G(d, q)$-modules, where $d$ runs through $\Delta$. (ii) the set of isomorphism types of all irreducible $\mathbb{F}(C_{d_0} \times C_q)$-modules.

**Proof.** Suppose $N$ is a normal subgroup of $G(m, q)$. If $N$ contains the commutator subgroup $G(m, q)'$ then $G(m, q)/N$ is abelian; otherwise, $N$ is contained in $M$, so $G(m, q)/N \cong G(d, q)$ for some $d$ in $\Delta$. On the other hand, for each $d$ in $\Delta$ there exists a unique normal subgroup $N$ such that $G(d, q) \cong G(m, q)/N$. Since $G(m, q)' = \langle x^{r-1} \rangle$ it follows easily that $G(m, q)/G(m, q)' \cong C_{d_0} \times C_q$, and hence the theorem is proven. □

**References**


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