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SOME FIXED POINT THEOREMS OF EXPANSION MAPPINGS

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1. Introduction.

Initially, the concept of 2-metric spaces has been investigated by Gähler in a series of papers [3], [4] and [5] and has been developed extensively by Gähler and many other researchers. On the other hand, the several authors have studies the aspects of fixed point theory for several types of contractive mappings in the setting of the 2-metric spaces. Recently, Chang-Kang [2] and Kang-Chang-Ryu [6] proved fixed point theorems of expansive mappings which correspond some contractive mappings in 2-metric spaces. In 1992, Chang [1] showed that suitable conditions are necessary in their results.

In this paper, we prove some common fixed point theorems of expansive mappings. Some of our results improve the results obtained by Chang-Kang [2] and Kang-Chang-Ryu [6].

2. Preliminaries

From Gähler [3] and White [8], we have the following definitions.

DEFINITION 2.1. A *2-metric space* is a set X with a real-valued function d on $X \times X \times X$ satisfying the following conditions:

(M_1) For distinct points x, y in X, there exists a point z in X such that $d(x,y,z) \neq 0$,

(M_2) $d(x,y,z) = 0$ if at least two of x, y, z are equal,

(M_3) $d(x,y,z) = d(x,z,y) = d(y,z,x)$,

(M_4) $d(x,y,z) \leq d(x,y,u) + d(x,u,z) + d(u,y,z)$ for all x, y, z, u in X.

The function d is called a *2-metric* for the space X and (X,d) denotes a 2-metric space. It has been shown by Gähler [3] that a 2-metric d is non-negative.

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DEFINITION 2.2. A sequence $\{x_n\}$ in a 2-metric space (X, d) is said to be *convergent* to a point x in X if $\lim_n d(x_n, x, a) = 0$ for all a in X.

DEFINITION 2.3. A sequence $\{x_n\}$ in a 2-metric space (X, d) is said to be a *Cauchy sequence* if $\lim_{m,n} d(x_m, x_n, a) = 0$ for all a in X. A 2-metric space (X, d) is said to be *complete* if every Cauchy sequence in X is convergent.

DEFINITION 2.4. A mapping S from a 2-metric space (X, d) into itself is said to be *sequentially continuous* at a point x in X if for every sequence $\{x_n\}$ in X such that $\lim_n d(x_n, x, a) = 0$ for all a in X, $\lim_n d(Sx_n, Sx, a) = 0$ for all a in X.

Throughout this paper, let $\mathcal{F}$ be the family of mappings such that for each $\phi \in \mathcal{F}$, $\phi : [0, \infty) \to [0, \infty)$ is upper semi-continuous from the right and non-decreasing in each coordinate variable with $\phi(t) < t$ for all $t > 0$.

We also need the following Lemma due to Matkoski [7] in the proof of our main theorems.

LEMMA. *If $\phi(t) < t$ for every $t > 0$, then $\lim_n \phi^n(t) = 0$, where $\phi^n(t)$ denotes the composition of $\phi(t)$ with n-times.*

3. The Main Theorems

Now, we prove some common fixed point theorems which is motivated by the expansive condition used by Kang-Chang-Ryu [6].

THEOREM 3.1. *Let S and T be mappings from a 2-metric space (X, d) into itself such that $S(X) \subseteq S^2(X)$, $S(X) \subseteq TS(X)$ and $S(X)$ is complete. Suppose that there exists $\phi \in \mathcal{F}$ such that for each x, y and a in X, at least one of the following conditions holds:*

(i) $$\phi\big(d(S^2x, TSy, a)\big) \geq d(Sx, Sy, a).$$

(ii) $$\phi\big(d(S^2x, TSy, a)\big) \geq \frac{1}{2}[d(Sx, Sy, a) + d(TSy, Sy, a)].$$

(iii) $$\phi\big(d(S^2x, TSy, a)\big) \geq \frac{1}{2}[d(Sx, S^2x, a) + d(Sx, Sy, a)].$$

(iv) $$\phi\big(d(S^2x, TSy, a)\big) \geq \frac{1}{3}[d(Sx, S^2x, a) + d(TSy, Sy, a) + d(Sx, Sy, a)].$$

Then either S or T has a fixed point, or S and T have a common fixed point.

Proof. Let x_0 be an arbitrary point X. Since $S(X) \subseteq S^2(X)$ and $S(X) \subseteq TS(X)$, we have for $x_0 \in X$, there exists a point x_1 in X such that $S^2 x_1 = Sx_0 = y_0$, say, and for this point x_1, there exists a point x_2 in X such that $TSx_2 = Sx_1 = y_1$, say. Inductively, we can define a sequence $\{y_n\}$ in $S(X)$ such that

$$S^2 x_{2n+1} = Sx_{2n} = y_{2n} \text{ and } TSx_{2n+2} = Sx_{2n+1} = y_{2n+1}.$$

It is easy to show that, for each of the inequalities (i)~(iv), that we have $\phi\big(d(y_{2n}, y_{2n+1}, a)\big) \geq d(y_{2n+1}, y_{2n+2}, a)$. Then one can show that $\phi\big(d(y_{2n+1}, y_{2n+2}, a)\big) \geq d(y_{2n+2}, y_{2n+3}, a)$, hence for arbitrary n,

$$\phi\big(d(y_n, y_{n+1}, a)\big) \geq d(y_{n+1}, y_{n+2}, a)$$

for all a in X. Now, if $y_{2n} = y_{2n+1}$ for any n, one has that y_{2n} is a fixed point of S, from the definition $\{y_n\}$. It then follows that, also, $y_{2n+1} = y_{2n+2}$, which implies that y_{2n} is also a fixed point of T.

For an arbitrary n, we have

$$d(y_n, y_{n+1}, a) \leq \phi\big(d(y_{n-1}, y_n, a)\big) \leq \cdots \leq \phi^n\big(d(y_0, y_1, a)\big)$$

for all a in X. By Lemma, $\lim_n d(y_n, y_{n+1}, a) = 0$ for all a in X.

Now, using the technique of Kang-Chang-Ryu [6], one would prove that $\{y_n\}$ a Cauchy sequence and it converges to some point y in $S(X)$. Consequently, the subsequences $\{y_{2n}\}$, $\{y_{2n+1}\}$ and $\{y_{2n+2}\}$ converge to y. Let $y = S^2 u$ and $y = TSv$ for some u and v in X, respectively. From inequalities (i)~(iv), it follows that at least one of the following inequalities must be true for an infinite number of values of n: for all a in X,

$$\phi\big(d(y_{2n}, y, a)\big) \geq d(Sx_{2n+1}, Sv, a)$$

$$\phi\big(d(y_{2n}, y, a)\big) \geq \frac{1}{2}[d(Sx_{2n+1}, Sv, a) + d(TSv, Sv, a)]$$

$$\phi\big(d(y_{2n}, y, a)\big) \geq \frac{1}{2}[d(Sx_{2n+1}, S^2 x_{2n+1}, a) + d(Sx_{2n+1}, Sv, a)]$$

$$\phi\big(d(y_{2n}, y, a)\big) \geq \frac{1}{3}[d(Sx_{2n+1}, S^2 x_{2n+1}, a) + d(TSv, Sv, a) + d(Sx_{2n+1}, Sv, a)]$$

Taking the limit as $n \to \infty$ in each case yields $y = Sv$. A similar argument applies to proving that $y = Su$. Therefore, y is a common fixed point of S and T. This completes the proof.

THEOREM 3.2. *Let S and T be sequentially continuous mappings from a 2-metric space (X, d) into itself such that $S(X) \subseteq S^2(X)$, $S(X) \subseteq TS(X)$ and $S(X)$ is complete. Suppose that there exists $\phi \in \mathcal{F}$ such that*

$$\phi\big(d(S^2x, TSy, a)\big) \geq \min\{d(Sx, S^2x, a), d(TSy, Sy, a), d(Sx, Sy, a)\}$$

for all x, y and a in X, where $\sum_{n=0}^{\infty} \phi^n(t) < \infty$ for all $t > 0$.
Then S or T has a fixed point or S and T have a common fixed point.

Proof. Define a sequence $\{y_n\}$ as in Theorem 3.1. If $y_n = y_{n+1}$ for any n, then S or T has a fixed point.

It is easy to show that, for each of the given inequality, that we have $\phi\big(d(y_{2n}, y_{2n+1}, a)\big) \geq d(y_{2n+1}, y_{2n+2}, a)$. Then one can show that $\phi\big(d(y_{2n+1}, y_{2n+2}, a)\big) \geq d(y_{2n+2}, y_{2n+3}, a)$, hence for arbitrary n,

$$\phi\big(d(y_n, y_{n+1}, a)\big) \geq d(y_{n+1}, y_{n+2}, a)$$

for all a in X. As in Theorem 3.1, we have that $\lim_n d(y_n, y_{n+1}, a) = 0$ for all a in X. From $\phi(0) = 0$, we have for every non-negative integer m,

$$\begin{aligned}
d(y_0, y_1, y_m) &\leq d(y_0, y_1, y_{m-1}) + d(y_m, y_{m-1}, y_0) + d(y_m, y_{m-1}, y_1) \\
&\leq d(y_0, y_1, y_{m-1}) + \phi^{m-1}\big(d(y_1, y_0, y_0) + d(y_1 y_0, y_1)\big) \\
&= d(y_0, y_1, y_{m-1}) \leq d(y_0, y_1, y_{m-2}) \leq \cdots \\
&\leq d(y_0, y_1, y_1) = 0.
\end{aligned}$$

Therefore, we obtain $d(y_n, y_{n+1}, y_m) = 0$. For arbitrary non-negative integers i, j and k $(0 \leq i < j < k)$,

$$\begin{aligned}
d(y_i, y_j, y_k) &\leq d(y_i, y_j, y_{i+1}) + d(y_i, y_{i+1}, y_k) + d(y_{i+1}, y_j, y_k) \\
&= d(y_{i+1}, y_j, y_k) \\
&\leq \cdots \leq d(y_{j-1}, y_j, y_k) = 0.
\end{aligned}$$

Thus, $d(y_i, y_j, y_k) = 0$. For any $m < n$, for all a in X,

$$\begin{aligned} d(y_m, y_n, a) &\le d(y_m, y_n, y_{m+1}) + d(y_m, y_{m+1}, a) + d(y_{m+1}, y_n, a) \\ &= d(y_m, y_{m+1}, a) + d(y_{m+1}, y_n, a) \\ &\le d(y_m, y_{m+1}, a) + d(y_{m+1}, y_{m+2}, a) + \cdots \\ &\qquad + d(y_{n-1}, y_n, a) \\ &\le \phi^m\big(d(y_0, y_1, a)\big) + \cdots + \phi^{n-1}\big(d(y_0, y_1, a)\big). \end{aligned}$$

From $\sum_{n=0}^{\infty} \phi^n(t) < \infty$ for all $t > 0$, it follows that $\{y_n\}$ is a Cauchy sequence and it converges to some point y in $S(X)$. Consequently, $\{y_{2n}\}$, $\{y_{2n+1}\}$ and $\{y_{2n+2}\}$ converge to y. By the sequentially continuity of S and T,

$$\left.\begin{aligned} S^2x_{2n+1} &= Sy_{2n+1} = y_{2n} \to Sy \\ TSx_{2n+2} &= Ty_{2n+2} = y_{2n+1} \to Ty \end{aligned}\right. \quad \text{as } n \to \infty.$$

Thus, S and T have a common fixed point.

COROLLARY 3.3. (1) *Let S and T be mappings from a 2-metric space (X, d) into itself such that $S(X) \subseteq S^2(X)$, $S(X) \subseteq TS(X)$ and $S(X)$ is complete. Suppose that there exists real numbers $h > 1$ such that for each x, y and a in X, at least one of the following conditions holds:*

$$d(S^2x, TSy, a) \ge h\, d(Sx, Sy, a).$$

$$d(S^2x, TSy, a) \ge \frac{h}{2}[d(Sx, Sy, a) + d(TSy, y, a)].$$

$$d(S^2x, TSy, a) \ge \frac{h}{2}[d(Sx, S^2x, a) + d(Sx, Sy, a)].$$

$$d(S^2x, TSy, a) \ge \frac{h}{3}[d(Sx, S^2x, a) + d(TSy, Sy, a) + d(Sx, Sy, a)].$$

Then either S or T has a fixed point, or S and T have a common fixed point.

(2) *Let S and T be sequentially continuous mappings from a 2-metric space (X, d) into itself such that $S(X) \subseteq S^2(X)$, $S(X) \subseteq TS(X)$ and $S(X)$ is complete. Suppose that there exists $h > 1$ such that*

$$d(S^2x, TSy, a) \ge h \min\{d(Sx, S^2x, a), d(TSy, Sy, a), d(Sx, Sy, a)\}$$

for all x, y and a in X.
Then S or T has a fixed point or S and T have a common fixed point.

Proof. For $\phi \in \mathcal{F}$, we define $\phi : [0, \infty) \to [0, \infty)$ by $\phi(t) = \frac{1}{h}\, t$, where $h > 1$. From Theorem 3.1 and 3.2, we obtain (1) and (2), respectively.

THEOREM 3.4. *Let S and T be mappings from a 2-metric space (X, d) into itself such that $S(X) \subseteq S^2(X)$, $S(X) \subseteq TS(X)$ and $S(X)$ is complete. Suppose that there exists non-negative real numbers $\alpha < 1$, $\beta < 1$ and γ $(\alpha + \beta + \gamma > 1)$ such that*

$$d(S^2x, TSy, a) \geq \alpha\, d(Sx, S^2x, a) + \beta\, d(TSy, Sy, a) + \gamma\, d(Sx, Sy, a)$$

for all x, y and a in X.
Then S and T have a common fixed point.

Proof. Define a sequence $\{y_n\}$ as in Theorem 3.1. Suppose that $y_{2n} = y_{2n+1}$ for some n. Then, for all a in X,

$$\begin{aligned} d(y_{2n}, y_{2n+1}, a) &= d(S^2x_{2n+1}, TSx_{2n+2}, a) \\ &\geq \alpha\, d(Sx_{2n+1}, S^2x_{2n+1}, a) + \beta\, d(TSx_{2n+2}, Sx_{2n+2}, a) \\ &\quad + \gamma\, d(Sx_{2n+1}, Sx_{2n+2}, a) \\ &= \alpha\, d(y_{2n}, y_{2n+1}, a) + \beta\, d(y_{2n+1}, y_{2n+2}, a) \\ &\quad + \gamma\, d(y_{2n+1}, y_{2n+2}, a), \end{aligned}$$

that is, $d(y_{2n}, y_{2n+1}, a) \geq \left(\frac{\beta+\gamma}{1-\alpha}\right) d(y_{2n+1}, y_{2n+2}, a)$ which says that $y_{2n+1} = y_{2n+2}$ since $\beta + \gamma \neq 0$. Thus, y_{2n} is a common fixed point of S and T. Similarly, $y_{2n+1} = y_{2n+2}$ gives that y_{2n+1} is a common fixed point of S and T.

Now, suppose that $y_n \neq y_{n+1}$ for each n. Then, for all a in X,

$$\begin{aligned} d(y_{2n}, y_{2n+1}, a) &= d(S^2x_{2n+1}, TSx_{2n+2}, a) \\ &\geq \alpha\, d(y_{2n}, y_{2n+1}, a) + \beta\, d(y_{2n+1}, y_{2n+2}, a) \\ &\quad + \gamma\, d(y_{2n+1}, y_{2n+2}, a). \end{aligned}$$

Thus, we have

$$d(y_{2n+1}, y_{2n+2}, a) \leq p_1\, d(y_{2n}, y_{2n+1}, a), \quad \text{where } p_1 = \frac{1-\alpha}{\beta+\gamma} < 1$$

for all a in X. Similarly, we have, for all a in X,

$$d(y_{2n+2}, y_{2n+3}, a) \le p_2 d(y_{2n+1}, y_{2n+2}, a), \quad \text{where } p_2 = \frac{1-\beta}{\alpha+\gamma} < 1.$$

Putting $p = \max\{p_1, p_2\}$, we have, for all a in X,

$$d(y_n, y_{n+1}, a) \le p\, d(y_{n-1}, y_n, a).$$

Therefore, $\{y_n\}$ is a Cauchy sequence and it converges to some point y in $S(X)$. Consequently, the subsequences $\{y_{2n}\}$, $\{y_{2n+1}\}$ and $\{y_{2n+2}\}$ converge to y. Let $y = S^2u$ and $y = TSv$ for some u and v in X, respectively. Then, for all a in X,

$$d(y_{2n}, y, a) = d(S^2 x_{2n+1}, TSv, a).$$

Letting $n \to \infty$, we have $0 \ge (\beta+\gamma)d(y, Sv, a)$ for all a in X, so that $y = Sv$. Similarly, $y = Su$. Therefore, S and T have a common fixed point.

REMARK. Our results improve several results of Chang-Kang [2] and Kang-Chang-Ryu [6] (see, Chang [1]). Furthermore, we have used non-surjective mappings.

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