THE GENERALIZED HURWITZ ZETA FUNCTION

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1. Introduction and Preliminaries

In [2], Barnes defines the generalized Hurwitz zeta function as follows:

\[ \zeta(s, a, w) = \sum_{n=0}^{\infty} \frac{1}{(a + nw)^s} \]

which can be continued analytically to the whole s-plane except a simple pole at \( s = 1 \). For all values \( s, a \) and \( w \) with \( \text{Re}(\frac{a}{w}) > 0 \) and \( \text{Re}(w) > 0 \), we can represent by the contour integral

\[
(1) \quad \zeta(s, a, w) = \frac{\Gamma(1-s)}{2\pi} \int_{C} \frac{e^{-az}}{(1-e^{-wz})(-z)^{s-1}} dz,
\]

where the contour \( C \) is a loop around the axis of \( w^{-1} \) round the origin from \( +\infty \) to \( +\infty \) as in Fig.1, and \(( -z)^{s-1} \) being equal to \( e^{(s-1)\log(-z)} \), where the real value of the logarithm is to be taken when \( z \) is negative. The contour must not embrace any zeroes of \( 1 - e^{-wz} \) except the origin. It is clear that \( \zeta(s, a, 1) = \zeta(s, a) \) where \( \zeta(s, a) \) is the well-known Hurwitz zeta function and \( \zeta(s, 1) = \zeta(s) \), where \( \zeta(s) \) is the Riemann zeta function. In this note we deduce some properties of \( \zeta(s, a, w) \) and generalize some results of \( \zeta(s, a) \).

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2. Some properties of $\zeta(s, a, w)$

We can express $\zeta(s, a, w)$ as a series of circular functions.

**Theorem 2.1.** We have, for $\text{Re}(\frac{a}{w}) > 0$ and $\text{Re}(w) > 0$,

$$\zeta(s, a, w) = \Gamma(1 - s)w^{-s}2^{s-1}\pi^{s-1}$$

$$\times \sum_{m=1}^{\infty} m^{s-1}2\cos\left[\frac{\pi}{2}(s - 1) + \frac{a}{w}2m\pi\right].$$

**Proof.** For the proof we make use of the contour $\Gamma_n = C_n - C$ in Fig. 2.

We assume that the square $C_n$ contains the points $\pm\frac{2m\pi}{w}, m = 1, 2, \cdots, n$ and the contour $C$ reduce the contour to a straight line from $+\infty$ to $\epsilon,$
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A small circle of radius $\varepsilon$ round the origin, and a straight line back from $+\varepsilon$ to $+\infty$. Since the contour $\Gamma_n$ has winding number one about the points $\pm \frac{2m\pi i}{w}$ with $m = 1, \ldots, n$. At these points $z = \pm \frac{2m\pi i}{w}$, the function $\frac{e^{(w-a)z}}{e^{wz} - 1}(-z)^{s-1}$ has simple poles with residues $(\frac{e}{w}) \frac{\pi i}{2} (\mp \frac{2m\pi i}{w})^{s-1}$. It follows that

$$\frac{1}{2\pi i} \int_{\Gamma_n} \frac{e^{(w-a)z}}{e^{wz} - 1}(-z)^{s-1}$$

$$= \sum_{m=1}^{n} w^{-s}(2m\pi)^{s-1}2\cos\left(\frac{\pi}{2}(s-1)\right)[e^{-\frac{a}{w}2m\pi i} + e^{\frac{a}{w}2m\pi i}].$$

If $\text{Re}(s) < 0$, the integral over $C_n$ will tend to zero as $n \to \infty$. Therefore, the integral over $C_n - C$ will tend to the integral over $-C$ as $n \to \infty$.

Hence,

$$\frac{1}{2\pi i} \int_{-C} \frac{e^{(w-a)z}}{e^{wz} - 1}(-z)^{s-1}dz$$

$$= w^{-s}2^{s-1}\pi^{s-1}\sum_{m=1}^{\infty} m^{s-1}[2\cos\left(\frac{\pi}{2}(s-1)\right)\cos\left(\frac{a}{w}2m\pi\right)$$

$$- 2\sin\left(\frac{\pi}{2}(s-1)\right)\sin\left(\frac{a}{w}2m\pi\right)].$$

We can thus have the desired expression (2)

**Corollary 2.2.** We have, letting $w = a$ in (2), our generalized Hurwitz zeta function can be reduced to the Riemann zeta function as follows:

$$\zeta(s, w, w) = \frac{\zeta(s)}{w^s}$$

Now, we define the generalized Bernoulli polynomials and numbers and then can evaluate $\zeta(s, a, w)$ for some special values of $s$. 
DEFINITION 2.3. The generalized Bernoulli polynomials $B_l(x, w)$ and numbers $B_l(0, w)$ are defined by respectively, for any complex number $x$,

$$\frac{ze^{xz}}{e^{wz}} = \sum_{l=0}^{\infty} \frac{B_l(x, w)}{l!} z^l, \quad |z| < \frac{2\pi}{|w|},$$

$$\frac{z}{e^{wz} - 1} = \sum_{l=0}^{\infty} \frac{B_l(0, w)}{l!} z^l, \quad |z| < \frac{2\pi}{|w|}.$$

Note that $B_l(x, 1) = B_l(x)$ and $B_l(0, 1) = B_l$, where $B_l(x)$ and $B_l$ are Bernoulli polynomials and numbers.

THEOREM 2.4. We have, for every nonnegative integers $l$,

$$B_l(x, w) = (-1)^l B_l(w - x, w).$$

Proof. For $|z| < \frac{2\pi}{|w|}$, we have

$$\frac{ze^{(w-x)z}}{e^{wz} - 1} = \sum_{l=0}^{\infty} \frac{B_l(w - x, w)}{l!} z^l.$$

Replacing $z$ by $-z$ in the resulting identity leads to

$$\frac{(-z)e^{(x-w)z}}{e^{-wz} - 1} = \sum_{l=0}^{\infty} \frac{B_l(w - x, w)}{l!} (-z)^l.$$

On the other hand

$$\frac{(-z)e^{(x-w)z}}{e^{-wz} - 1} = \frac{ze^{wz}}{e^{wz} - 1} = \sum_{l=0}^{\infty} \frac{B_l(x, w)}{l!} z^l.$$

Equating coefficients of $z^l$, we obtain the desired result.
THEOREM 2.5. For every integer $l \geq 0$, we have

$$\zeta(-l, a, w) = (-1)^l \frac{B_{l+1}(w - a, w)}{l + 1}.$$  

Proof. From (1), we have $\zeta(s, a, w) = \Gamma(1 - s) I(s, a, w)$, where

$$I(s, a, w) = -\frac{1}{2\pi i} \int_C \frac{e^{-az}}{1 - e^{-wz}} (-z)^{s-1} dz$$

Hence,

$$I(-l, a, w) = -\text{Res}_{z=0} \frac{e^{-az}}{1 - e^{-wz}} (-z)^{-l-1}$$

$$= \frac{(-1)^l}{(l+1)!} B_{l+1}(w - a, w).$$

From Theorems 2.4 and 2.5, we have the following.

COROLLARY 2.6. For every integer $l \geq 0$, we have

$$\zeta(-l, a, w) = -\frac{1}{l+1} B_{l+1}(a, w)$$

THEOREM 2.7. From Theorems 2.1 and Corollary 2.6, we can express Fourier sine and cosine series of $B_{2k+1}(a, w)$ and $B_{2k}(a, w)$:

$$B_{2k+1}(a, w) = (-1)^k (2k+1)! w^{2k} (2\pi)^{-2k-1} \sum_{m=1}^{\infty} 2m^{-2k-1} \sin\left(\frac{a}{w} 2m\pi\right)$$

$$B_{2k}(a, w) = (-1)^{k+1} (2k)! w^{2k-1} (2\pi)^{-2k} \sum_{m=1}^{\infty} 2m^{-2k} \cos\left(\frac{a}{w} 2m\pi\right)$$

for $k = 1, 2, 3, \ldots$

Proof. From (2), letting $s = -2k$, $k = 1, 2, \ldots$ Then

$$\zeta(-2k, a, w) = \Gamma(2k + 1) w^{2k} 2^{-2k-1} \pi^{-2k-1}$$

$$\times \sum_{m=1}^{\infty} 2m^{-2k-1} (-1)^{k+1} \sin\left(\frac{a}{w} 2m\pi\right).$$
And from Corollary 2.6, we get the following

\[(4) \quad \zeta(-2k, a, w) = -\frac{1}{2k + 1} B_{2k+1}(a, w).\]

Comparing (3) and (4) yields the desired results.

Letting \(a = 0\) and \(w = 1\) in Theorem 2.7, we have the followings.

**Corollary 2.8.** For \(k = 1, 2, 3, \ldots\),

\[
B_{2k+1}(0, 1) = B_{2k+1} = 0
\]

\[
B_{2k}(0, 1) = B_{2k} = (-1)^{k+1}(2k)!/(2\pi)^{-2k}2\zeta(2k).
\]

where the above equalities are well known result (see [4,p.332]).

**References**


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