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THE GENERALIZED HURWITZ ZETA FUNCTION

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1. Introduction and Preliminaries

In [2], Barnes defines the generalized Hurwitz zeta function as follows :

$$\zeta(s,a,w)=\sum_{n=0}^{\infty}\frac{1}{(a+nw)^s},$$

which can be continued analytically to the whole s-plaine except a simple pole at $s=1$. For all values s, a and w with $\mathrm{Re}(\frac{a}{w})>0$ and $\mathrm{Re}(w)>0$, we can represent by the contour integral

$$(1)\qquad \zeta(s,a,w)=\frac{i\Gamma(1-s)}{2\pi}\int_C\frac{e^{-az}}{(1-e^{-wz})}(-z)^{s-1}dz,$$

where the contour C ic a loop around the axis of w^{-1} round the origin from $+\infty$ to $+\infty$ as in Fig.1, and $(-z)^{s-1}$ being equal to $e^{(s-1)\log(-z)}$, where the real value of the logarithm is to be taken when z is negative. The contour must not embrace any zeroes of $1-e^{-wz}$ except the origin. It is clear that $\zeta(s,a,1)=\zeta(s,a)$ where $\zeta(s,a)$ is the well-known Hurwitz zeta function and $\zeta(s,1)=\zeta(s)$, where $\zeta(s)$ is the Riemann zeta function. In this note we deduce some properties of $\zeta(s,a,w)$ and generalize some results of $\zeta(s,a)$.

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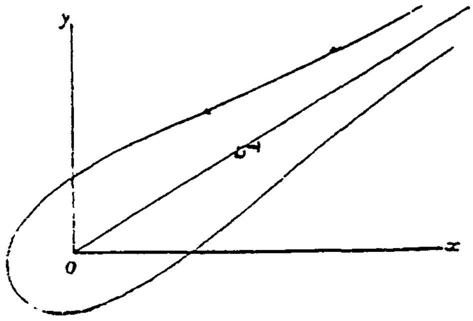

Fig. 1.

2. Some properties of $\zeta(s,a,w)$

We can express $\zeta(s,a,w)$ as a series of circular functions.

THEOREM 2.1. *We have , for* $Re(\frac{a}{w})>0$ *and* $Re(w)>0$,

$$\zeta(s,a,w)=\Gamma(1-s)w^{-s}2^{s-1}\pi^{s-1}$$

$$\times\sum_{m=1}^{\infty}m^{s-1}2\cos[\frac{\pi}{2}(s-1)+\frac{a}{w}2m\pi]. \tag{2}$$

Proof. For the proof we make use of the contour $\Gamma_n=C_n-C$ in Fig.2.

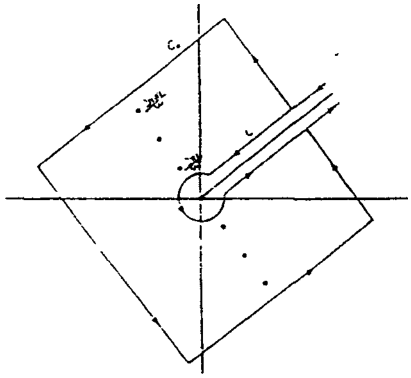

Fig. 2.

We assume that the square C_n contains the points $\pm\frac{2m\pi}{w}i$, $m=1,2,\cdots,n$ and the contour C reduce the contour to a straight line from $+\infty$ to ϵ, a small circle of radius ϵ round the origin, and a straight line back from $+\epsilon$ to $+\infty$. Since the contour Γ_n has winding number one about the points $\pm\frac{2m\pi i}{w}$ with $m = 1, \dots, n$. At these points $z = \pm\frac{2m\pi i}{w}$, the function $\frac{e^{(w-a)z}}{e^{wz}-1}(-z)^{s-1}$ has simple poles with residues $(\frac{e}{w})^{\frac{a}{w}}(\mp\frac{2m\pi i}{w})^{s-1}$. It follows that

$$\frac{1}{2\pi i}\int_{\Gamma_n}\frac{e^{(w-a)z}}{e^{wz}-1}(-z)^{s-1}$$
$$= \sum_{m=1}^{n} w^{-s}(2m\pi)^{s-1}2\cos(\frac{\pi}{2}(s-1))[e^{-\frac{a}{w}2m\pi i} + e^{\frac{a}{w}2m\pi i}].$$

If $\mathrm{Re}(s) < 0$, the integral over C_n will tend to zero as $n \to \infty$ Therefore, the integral over $C_n - C$ will tend to the integral over $-C$ as $n \to \infty$.

Hence,

$$\frac{1}{2\pi i}\int_{-C}\frac{e^{(w-a)z}}{e^{wz}-1}(-z)^{s-1}dz$$
$$= w^{-s}2^{s-1}\pi^{s-1}\sum_{m=1}^{\infty} m^{s-1}[2\cos(\frac{\pi}{2}(s-1))\cos(\frac{a}{w}2m\pi)$$
$$- 2\sin(\frac{\pi}{2}(s-1))\sin(\frac{a}{w}2m\pi)].$$

We can thus have the desired expression (2)

COROLLARY 2.2. *We have , letting $w = a$ in (2), our generalized Hurwitz zeta function can be reduced to the Riemann zeta function as follows:*

$$\zeta(s, w, w) = \frac{\zeta(s)}{w^s}$$

Now, we define the generalized Bernoulli polynomials and numbers and then can evaluate $\zeta(s, a, w)$ for some special values of s.

DEFINITION 2.3. *The generlaized Bernoulli polynomials* $B_l(x,w)$ *and numbers* $B_l(0,w)$ *are defined by respectively , for any complex number* x,

$$\frac{ze^{xz}}{e^{wz}} = \sum_{l=0}^{\infty} \frac{B_l(x,w)}{l!} z^l, \qquad |z| < \frac{2\pi}{|w|},$$
$$\frac{z}{e^{wz}-1} = \sum_{l=0}^{\infty} \frac{B_l(0,w)}{l!} z^l, \qquad |z| < \frac{2\pi}{|w|}.$$

Note that $B_l(x,1) = B_l(x)$ and $B_l(0,1) = B_l$, where $B_l(x)$ and B_l are Bernoulli polynomials and numbers.

THEOREM 2.4. *We have, for every nonnegative integers* l,

$$B_l(x,w) = (-1)^l B_l(w-x,w).$$

Proof. For $|z| < \frac{2\pi}{w}$, we have

$$\frac{ze^{(w-x)z}}{e^{wz}-1} = \sum_{l=0}^{\infty} \frac{B_l(w-x,w)}{l!} z^l.$$

Replacing z by $-z$ in the resulting identity leades to

$$\frac{(-z)e^{(x-w)z}}{e^{-wz}-1} = \sum_{l=0}^{\infty} \frac{B_l(w-x,w)}{l!} (-z)^l.$$

On the other hand

$$\frac{(-z)e^{(x-w)z}}{e^{-wz}-1} = \frac{ze^{xz}}{e^{wz}-1} = \sum_{l=0}^{\infty} \frac{B_l(x,w)}{l!} z^l.$$

Equating coefficients of z^l , we obtain the desired result.

THEOREM 2.5. *For every integer $l \geq 0$, we have*

$$\zeta(-l, a, w) = (-1)^l \frac{B_{l+1}(w-a, w)}{l+1}.$$

Proof. From (1), we have $\zeta(s, a, w) = \Gamma(1-s) I(s, a, w)$, where

$$I(s, a, w) = -\frac{1}{2\pi i} \int_C \frac{e^{-az}}{1-e^{-wz}} (-z)^{s-1} dz$$

Hence,

$$\begin{aligned} I(-l, a, w) &= -\text{Res}_{z=0} \frac{e^{-az}}{1-e^{-wz}} (-z)^{-l-1} \\ &= \frac{(-1)^l}{(l+1)!} B_{l+1}(w-a, w). \end{aligned}$$

From Theorems 2.4 and 2.5, we have the following.

COROLLARY 2.6 *For every integer $l \geq 0$, we have*

$$\zeta(-l, a, w) = -\frac{1}{l+1} B_{l+1}(a, w)$$

THEOREM 2.7. *From Theorems 2.1 and Corollary 2.6, we can express Fourier sine and cosine series of $B_{2k+1}(a, w)$ and $B_{2k}(a, w)$:*

$$B_{2k+1}(a, w) = (-1)^k (2k+1)! w^{2k} (2\pi)^{-2k-1} \sum_{m=1}^{\infty} 2m^{-2k-1} \sin(\frac{a}{w} 2m\pi)$$

$$B_{2k}(a, w) = (-1)^{k+1} (2k)! w^{2k-1} (2\pi)^{-2k} \sum_{m=1}^{\infty} 2m^{-2k} \cos(\frac{a}{w} 2m\pi)$$

for $k = 1, 2, 3, \ldots$

Proof. From (2), letting $s = -2k, k = 1, 2, \ldots$ Then

$$\begin{aligned} \zeta(-2k, a, w) &= \Gamma(2k+1) w^{2k} 2^{-2k-1} \pi^{-2k-1} \\ &\times \sum_{m=1}^{\infty} 2m^{-2k-1} (-1)^{k+1} \sin(\frac{a}{w} 2m\pi). \end{aligned} \tag{3}$$

And from Corollary 2.6, we get the following

$$\zeta(-2k, a, w) = -\frac{1}{2k+1} B_{2k+1}(a, w). \tag{4}$$

Comparing (3) and (4) yields the desired results.

Letting $a = 0$ and $w = 1$ in Theorem 2.7, we have the followings.

COROLLARY 2.8. *For* $k = 1, 2, 3, \dots,$

$$B_{2k+1}(0,1) = B_{2k+1} = 0$$
$$B_{2k}(0,1) = B_{2k} = (-1)^{k+1}(2k)!(2\pi)^{-2k} 2\zeta(2k).$$

where the above equalities are well known result (see [4,p.332]*).*

References

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