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ON BINDING SUBSETS

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1. Preliminaries

Many of our algebraic ideas and notations come from those of R.C. Lyndon and P.E. Schupp [2].The concept of a binding subset was introduced by H.C. Lyon [3]. Let W be a finite subset of cyclic elements of a finitely generated free group F. W is said to *bind* F if W is connected and not contained in any proper free factor of F. By the fact that W is connected we mean that the incidence graph $J(W)$ of W is connected with respect to all bases for F. This paper is concerned with some sufficient conditions for which a finite subset W binds F. Now we introduce *Whitehead*'s theorems and *Lyon*'s free factoring theorem without proofs which are crucial for further consideration:

THEOREM 1.1. [2] *Let $w_1, w_2, \ldots, w_t$ and $w'_1, w'_2, \ldots, w'_t$ be cyclic words in a free group F such that $w_i\alpha = w'_i$ $(i = 1, \ldots, t)$ for some $\alpha \in Aut(F)$. If $\sum |w'_i|$ is minimal among all $\sum |w_i\alpha'|$ for all $\alpha' \in Aut(F)$, then there exists a sequence of Whitehead automorphisms of F, say $\alpha_1, \ldots, \alpha_n$, such that*

$$\sum_{i=1}^{t} |w_i\alpha_1 \ldots \alpha_j| \leq \sum_{i=1}^{t} |w_i|$$

for all $0 < j < n$. with strict ineqality unless $\sum |w_i| = \sum |w'_i|$.

THEOREM 1.2. [2] *Let $W = \{w_1, \ldots, w_t\}$ be a finite subset of a free group F. If $\alpha \in Aut(F)$ with $|W\alpha| \leq |W|$, there exists a sequence of Whitehead automorphisms $\alpha_1, \ldots, \alpha_n$ such that*

$$|W\alpha_1 \ldots \alpha_j| \leq |W|$$

for $1 \leq j \leq n$

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THEOREM 1.3. [3] *Let W be a finite set of elements (respectively, cyclic elements) in the finitely generated free group F. There exists a decomposition $F = F_0 * F_1 * \ldots * F_n$ with $F_i \neq 1$ if $i \neq 0$, such that*

(1) $W \subset \cup_{i\neq 0} F_i$;
(2) $W_i = W \cap F_i$ *is non-empty and connected for* $1 \leq i \leq n$;
(3) *The stabilizer of W in the automorphism group (outer automorphism group) of F is the direct product of the stabilizers of the components W_i in the (outer) automorphism groups of the F_i, $1 \leq i \leq n$;*
(4) *If $F = G_1 * G_2$ and $W \subset G_1 \cup G_2$, then each of $W \cap G_1$ and $W \cap G_2$ is the union of certain of the components W_i. and the corresponding F_i are (conjugate to) free factors of G_1 and G_2, respectively;*
(5) *If Y is any basis for F such that $|W|_Y$ is minimal, then each F_i has a basis equal to (conjugate to) some $Y_i \subset Y$; and*
(6) *If Y is any basis for F such that $|W|_Y$ is minimal, then the number of basis elements used in so expressing W is minimal.*

2. Main Results

THEOREM 2.1. *Let W be a finite subset of cyclic elements of the finitely generated free group F with a basis X such that $|W|_X$ is minimal If W is connected with respect to the specific basis X and contains all basis elements of X, then for any basis Y for F, W is connected with respect to Y and not contained in any proper free factor of F. That is, W binds F.*

Proof.. Let Y be any basis for F. Assume that W is contained in some proper free factor of $F =< Y >$. Then the number of basis elements of Y used in so expressing W is less then the rank of F. By theorem 1.3, the number of basis elements of X used in expressin W is minimal and so less than the rank of F. This is impossible. Therefore, we conclude that W is not contained in any proper free factor of $F =< Y >$. If W is not connected with respect to Y, then we can decompose Y into two non-empty subsets Y_1 and Y_2 such that

$$W \subset < Y_1 > \cup < Y_2 >$$

Applying the assumption of the theorem to theorem 1.3, we have $n = 1$ and $F_0 = 1$ and there is only one connected component W itself with respect to Y. From (4) of theorem 1.3, each of $W \cap < Y_1 >$ and $W \cap < Y_2 >$ is the union of certain of connected components of W. Hence, one, say $W \cap < Y_1 >$, of the $W \cap < Y_i >$ $(i = 1, 2)$ must be equal to W. This means that $W \subset < Y_1 >$, which is impossible since Y_1 is a proper subset of Y. So W is connected with respect to Y.

Let F be a free group with basis X and W a set of cyclic elements of F. We call W *quadratic* over a subset X_0 of X if no element $w \in W$ contains any x from X (as x or x_{-1}) except $x \in X_0$ and W contains such x at most twice. We call W *strictly quadratic* over X_0 if, moreover, each $x \in X_0$ occures in W exactly twice. Note that if W is strictly quadratic and contains all the generators, there exactly two edges at each vertex in the star graph $\sum(W)$ of W, whence if W is finite, $\sum(W)$ is a union of disjoint cycles.

THEOREM 2.2. *Let W be a finite set of cyclic elements over a free group F with a basis X. If W is strictly quadratic and the star graph $\sum(W)$ is a cycle, then W binds F*

Proof.. Suppose that W is not minimal There exists a Whitehead automorphism $\sigma = (A, a)$ such that $|W\sigma| < |W|$; That is, for some $w \in W$, $|w\sigma| < |w|$. Now, we have

$$D(\sigma, w) \stackrel{\text{def}}{=} |w\sigma| - |w| = A \cdot A' - a \cdot X^{\pm 1},$$

where $A' = X_{\pm 1} - A$, $x \cdot y$ is the number of segments of one of the forms xy^{-1} and yx^{-1} in w and $A \cdot A' =$ the sum of $a \cdot b$ $(a \in A, b \in B)$. Since W is strictly connected, we have

$$A\dot{A}' < a \cdot X^{\pm 1} = 2.$$

However, since $\sum(W)$ is a cycle, the vertex set $X^{\pm 1}$ can not split into disjoint subsets A and A' with $A \cdot A' = 1$. Hence $A \cdot A' = 0$. This contradicts the assumption that $\sum(W)$ is a cycles. Hence, $|W|_X$ is minimal. Since $\sum(W)$ is a cycle, W is connected with respect to X and W must contain all generators. Therefore, W is connected. By theorem 2.1, W binds F.

REMARK. *The minimality of* $|W|_X$ *in theorem 2.1 is necessary.*

3. Example

The following example will demonstrate that the minimality of $|W|_X$ is necessary in theorem 2.1:

EXAMPLE. *Let* F *be a free group with a basis* $X = \{x_1, x_2, x_3, x_4\}$ *and*

$$w_1 = x_1$$

$$w_2 = x_1 x_2$$

$$w_3 = x_1 x_2 x_3$$

$$w_4 = x_3 x_4 x_3^{-1} x_4^{-1},$$

which are the presentations of simple closed curves on the boundary of a handlebody of genus 4 in Przytycki's wrong example [4]. *Even if* $W = \{w_1, w_2, w_3, w_4\}$ *is connected with respect to* X, *it does not bind* F. *We can find a geometric argument in* [1] *to show that Przytycki was incorrect. In fact,* $|W|_X$ *is not minimal as it turned out.*

If a finite set W of cyclic elements in a free group of finite rank is realized as a finite system of pairwise disjoint, simple closed curves on the boundary of a handlebody, the binding property of W can be checked in terms of a geometric algorithm [1].

4. 3-MANIFOLD GROUP

Each splitting of the fundamental group of a compact 3-manifold with incompressible boundary (possibly empty) as a free product is induced by the splitting of the manifold as a connected sum. If a group $G = < x_1, \dots, x_k; w_1, w_2 >$ with k ($k > 1$) generators and 2 relators is realized by a handlebody H_k together with two disjoint simple closed curves γ_1 and γ_2 on ∂H_k and has properties that

(1) $\{w_1, w_2\}$ binds F_k and
(2) neither w_1 nor w_2 binds F_k,

then $G = \pi_1((H_k)_{\gamma_1\gamma_2}) = \pi_1((H_k)^+_{\gamma_1\gamma_2})$, $(H_k)_{\gamma_1\gamma_2}$ is equal to D^3 or is equal to an irreducible 3-manifold with incompressible boundary and $(H_k)^+_{\gamma_1\gamma_2}$ is irreducible [4], where M_γ is the 3-manifold obtained from M by attaching a 2-handle along γ and M^+_γ is obtained from M_γ by attaching a 3-handle. If $(H_k)_{\gamma_1\gamma_2}$ is equal to D^3, then the group G is trivial. Now, we assume that $(H_k)_{\gamma_1\gamma_2}$ is a 3-manifold with incompressible boundary and $(H_k)^+_{\gamma_1\gamma_2}$ is irreducible. If G has a decomposition $G = G_1 * G_2$ as a free product, then we have

$$(H_k)^+_{\gamma_1\gamma_2} = M_1 \sharp M_2.$$

where $G_i = \pi_1(M_i)$, $i = 1, 2$. However, since $(H_k)^+_{\gamma_1\gamma_2}$ is irreducible (hence, prime), one of the M_i is a 3-sphere. This implies that one of the G_i must be trivial. Thus G can not be decomposed into non-trivial free product Now, we formulate this as a theorem:

THEOREM 3.1. *Let $G = < x_1, \ldots, x_k; w_1, w_2 >$ $(k \geq 2)$ is realized by a handlebody H_k together with two disjoint simple closed curves on the boundary of H_k. If $\{w_1, w_2\}$ binds $F_k = < x_1, \ldots, x_k >$ and none of w_1 and w_2 binds F_k, then G is indecomposible.*

To determine whether or not a finite set W binds a free group F, from theorem 2.1 it suffices to check if the set W is connected only with respect to a basis X with $|W|_X$ minimal and each element of X occurs in W. Therefore, theorem 1.1 and theorem 1.2 together with theorem 2.1 provide us an algorithm for determining whether or not W binds F; We need only apply Whitehead automorphisms succesively until $|W|$ is minimal. This procedure terminates after finite number of applications and we observe the situation at this point.

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