STUDY OF THE BEST DEFORMATION FOR EXTENDING HARDY SPACES AND ITS APPLICATIONS

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Let $D$ be an open unit disc, $T$ be the unit circle in the complex plane. The Hardy space $H^p$ ($0 < p < \infty$) consists of all functions holomorphic in $D$ for which

$$\|f\|_p = \begin{cases} \lim_{r \to 1} \left[ \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \right]^{\frac{1}{p}}, & 0 < p < \infty \\ \sup_{z \in D} |f(z)|, & p = \infty \end{cases}$$

is finite. If we define

$$M_p(f, r) = \left[ \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \right]^{\frac{1}{p}}, \quad 0 < p < \infty$$

$$M_\infty(f, r) = \sup\{|f(re^{i\theta})| : 0 \leq \theta \leq 2\pi\},$$

we can rewrite

$$\|f\|_p = \lim_{r \to 1} M_p(f, r), \quad 0 < p \leq \infty.$$

A function bounded and holomorphic in $D$ is said to be an inner function if its boundary values have modulus 1 almost everywhere. A Blaschke sequence is a (finite or infinite) sequence $\{a_n\}$ of complex numbers satisfying the conditions; $0 < |a_n| < 1$ and $\sum(1 - |a_n|)$ is finite. An important class of inner function is the Blaschke product. A Blaschke product $B(z)$ with zeros $\{a_n\}$ is a function defined by a formula;

$$B(z) = \prod \frac{|a_n|}{a_n} \frac{a_n - z}{1 - \overline{a}_n z}$$

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for a Blaschke sequence \( \{a_n\} \). The set of Blaschke products is uniformly dense in the set of inner function by the Frostman's theorem [9,10].

Let \( B^p \) \((0 < p < 1)\) denote the space of functions \( f(z) \) holomorphic in \( D \) for which

\[
\frac{1}{2\pi} \int_0^1 \int_0^{2\pi} |f(re^{i\theta})(1-r)^{\frac{1}{2}-2}d\theta dr
\]

is finite. If we use the quantity \( M_p(f,r) \), we can rewrite as following;

\[
||f||_{B^p} = \int_0^1 (1-r)^{\frac{1}{2}-2}M_1(f,r)dr.
\]

It turns out \( H^p \) is a subspace of \( B^p \), especially \( B^p = H^p \) for \( p = \frac{1}{2} \). It was found [3,12], \( H^p \) is dense in \( B^p \) and two spaces have the same continuous linear functionals. This makes it possible to identify \( B^p \) with the closure of \( H^p \) in its second dual of \( H^p \) [1,5]. Thus this deformation space \( B^p \) is in some respects nicer than \( H^p \) space.

Now we introduce the weighted Bergman space \( A^{p,q} \) [7,11] and develop some of properties on \( A^{p,q} \) space. If \( f(z) \) is holomorphic in \( D \) and \( 0 < p < 1 \) and \( q > 0 \) we define the weighted \( L^q \) norm by

\[
\int_0^1 \int_0^{2\pi} |f(re^{i\theta})|^q(1-r)^{\frac{1}{2}-2}d\theta dr.
\]

If it is finite we say \( f(z) \) belongs to \( A^{p,q} \). This space is more extended than \( B^p \), especially \( A^{p,q} \) is equal to \( B^p \) when \( q = 1 \), at that time there are many interesting results on it.

G. Caughram and L. Shields raised the question whether there exits a singular inner function whose derivative is in \( H^p \) \((p = \frac{1}{2})\). L. Duren, W. Romberg and L. Shields [6] proved that the derivative of every Blaschke product lies in \( B^p \) for all \( p \) \((0 < p < \frac{1}{2})\). In [16], W. Rudin showed that if the zeros \( \{a_n\} \) of a Blaschke product satisfying the condition

\[
\sum (1-|a_n|) \log \frac{1}{1-|a_n|} < \infty,
\]

then the derivative of \( B(z) \) lies in \( B^{\frac{1}{2}} \). Also, P. Ahern and D. Clark [2] proved that if \( \sum (1-|a_n|)^{\frac{1}{2}} < \infty \), and \( \sum (1-|a_n|)^{\frac{1}{2}} \log \frac{1}{1-|a_n|} = \)
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∞, then there is a Blaschke product \( B(z) \) with zeros \( \{a_n\} \) satisfying \( B'(z) \in B^{\frac{2}{3}} \) and \( B'(z) \in H^{\frac{1}{3}} \). D. Protas [15] generalized this property as followings that if the zeros \( \{a_n\} \) satisfy \( \sum (1 - |a_n|)^\alpha < \infty \) then \( B'(z) \in B^{\frac{1-\alpha}{2}} \) for some \( \alpha \) \((0 < \alpha < 1)\), and \( B'(z) \in H^{1-\alpha} \) for \( \alpha \) \((0 < \alpha < \frac{1}{2})\). We could not translate all the \( B^p \) results into \( A^{p,q} \) space.

In this paper, we find some results in deformations of \( H^p \) spaces and consider the relation between the distribution of \( B(z) \) and \( \hat{B}(z) \). There are several known conditions on the distribution of Blaschke sequences that imply the derivative of Blaschke products lies in the extended \( H^p \) space. The basic problem we consider is that of determining \( A^{p,q} \) spaces to which the derivative of \( B(z) \) belongs.

For typographical reasons we frequently omit the superscript \( p \) in writing \( \||f||_{B^p} \). We first prove followings.

**Proposition 1.** For each \( f \) in \( B^p \), the following inequality holds for constant \( K_p \) (\( : \) depend on \( p \)).

\[ |f(z)| \leq K_p ||f||_B (1 - r)^{-\frac{1}{p}}. \]

**Proof.** Let \( R < r < 1 \), then

\[
\|f\|_B \geq \int_R^1 (1 - r)^{\frac{1}{p} - 2} M_1(f, r) dr \\
\geq M_1(f, R)(\frac{1}{p} - 1)^{-1}(1 - R)^{\frac{1}{p} - 1}.
\]

Hence

\[ M_1(f, R) \leq (\frac{1}{p} - 1) ||f||_B (1 - R)^{1 - \frac{1}{p}}. \]

From this, the estimate follows by writing

\[ f(z) = \frac{1}{2\pi i} \int_{|\zeta| = R} \frac{f(\zeta)}{\zeta - z} d\zeta, \]

where \( R = \frac{1}{2} (1 + |z|) \).
LEMMA 2. For each $f \in B^p$, $f_\rho \to f$ in $B^p$-norm as $\rho \to 1$, where $f_\rho(z) = f(\rho z)$.

Proof. Given $f \in B^p$ and $\varepsilon > 0$, choose $r < 1$ such that

$$\int_0^1 (1 - r)^{1-2} M_1(f, r) dr \leq \varepsilon \quad \cdots (2.1).$$

Since $M_1(f, r)$ is an increasing function of $r$, (2.1) remains valid when $f$ is replaced by $f_\rho$. Now choose $\rho$ so close to 1 that $|f_\rho(z) - f(z)| < \varepsilon$ on $|z| \leq R$. Then

$$\int_0^R (1 - r)^{1-2} M_1(f_\rho - f, r) dr < \varepsilon \|1\|_B.$$

Combining this with (2.1), we have

$$\|f_\rho - f\|_B \leq \varepsilon \|1\|_B + 2\varepsilon,$$

so $f_\rho \to f$ in norm as $\rho \to 1$.

LEMMA 3. For each $f \in H^p$, $\|f\|_B \leq K_p \|f\|_p$.

Proof. The above statement means that $H^p \subset B^p$, and gives the norm inequality. Also, $H^p$ contains all functions holomorphic in a bigger disc, and such functions are dense in $B^p$ by Lemma 2.

If we use above statements, the following fact is satisfied.

THEOREM 4. Let $\varphi$ be in the dual $(B^p)^*$ of $B^p$ for $0 < p < 1$, then there is unique function $g$ such that

$$\varphi(f) = \lim_{r \to 1} \int_0^{2\pi} f(re^{i\theta})g(e^{-i\theta})d\theta, \quad f \in B^p,$$

where $g(z)$ is holomorphic in $D$ and continuous on $\overline{D}$. 
We consider some relations between the distribution of the zeros of the \( k \)-th derivative \( B^{(k)}(z) \) of Blaschke product and the behavior of its Taylor coefficients

\[
\hat{B}(z) = \frac{B^{(k)}(0)}{k!} \quad (k \geq 0).
\]

Let \( f(x) \) be defined in a closed interval \( I \) and let

\[
\omega(\delta) = \omega(\delta, f) = \sup |f(x_2) - f(x_1)|
\]

for \( x_1, x_2 \in I, \quad |x_2 - x_1| \leq \delta \). The function \( \omega(\delta) \) is called the modulus of continuity of \( f \). If \( I \) is finite, then \( f \) is continuous in \( I \) if and only if \( \omega(\delta) \to 0 \) as \( \delta \to 0 \). For some \( \alpha > 0 \), we have \( \omega(\delta) \leq c\delta^\alpha \), where \( c \) is independent of \( \delta \).

Recall that \( f(z) \) satisfies a Lipschitz condition of order \( n \) in \( D \) (denote \( f \in \Lambda_n \)) if and only if

\[
|f(z_1) - f(z_2)| \leq c|z_1 - z_2|^n
\]

for \( 0 < n \leq 1 \) where \( c \) is independent of \( z_1, z_2 \) and \( z_1, z_2 \in D \). Similarly, \( f \in \Lambda_n^* \) means that

\[
|f(z_1) - f(z_2)| = o(|z_1 - z_2|^n).
\]

It is obvious that functions in classes \( \Lambda_n, \Lambda_n^* \) are bounded and continuous. Only the case \( 0 < n < 1 \) is interesting: if \( n > 1 \), then \( \omega(\delta)/\delta \) tends to zero with \( \delta \), \( f'(x) \) exists and is zero everywhere, and \( f \) is a constant. The function \( f \) belongs to \( \Lambda_1 \) if and only if \( f \) is integral of a bounded function.

Here, we apply these properties to the Hardy space. We recall that the function \( f \in L^p(T) \) is in the class \( \Lambda_\alpha^p \), \( (0 < \alpha < 1, 1 \leq p < \infty) \) if its \( L^p \)-modulus of continuity

\[
\omega_p(\delta) = \sup_{|\theta| \leq \delta} \left| \int_T |f(\zeta e^{i\theta}) - f(\zeta)|^p d\zeta \right|^{1/p}
\]

satisfies the condition \( \omega_p(\delta) \leq c\delta^\alpha \). According to the Hardy and Littlewood theorem, the boundary values of a function \( \psi \) from the Hardy space \( H^p \) belong to the class \( \Lambda_\alpha^p \) if and only if

\[
\left[ \int_T |\psi'(r\zeta)|^p d\zeta \right]^{1/p} \leq c(1 - r)^{\alpha - 1}.
\]
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Newman and Shapiro [14] have proved that the Taylor coefficient of an inner function may have order $o(\frac{1}{k})$ only in the trivial case of finite Blaschke products. For all $B(z)$, whose zeros satisfy the Newman condition

$$\sup_{k \geq 0} \frac{(1 - |a_{k+1}|)}{(1 - |a_k|)} < 1,$$

J. Newman and S. Shapiro obtained the estimate

$$\hat{B}(k) = O\left(\frac{1}{k}\right),$$

where $\hat{B}(k)$ is the Taylor coefficients of $B(k)$.

**Theorem 5.** Let $B(z)$ be a Blaschke product and let $\{z_k\}$ be its zero, then the following statements are equivalent:

1. The sequence $\{z_k\}$ satisfies the Newman condition,
2. $\hat{B}(k) = O\left(\frac{1}{k}\right)$,
3. $\sum_{k \geq n} |\hat{B}(k)|^2 = O\left(\frac{1}{n}\right)$,
4. $B(z) \in \Lambda^p_1$ for some $1 < p < \infty$, and
5. $\int_T |B''(r\zeta)||d\zeta| \leq c(1 - r)^{-1}$ for some constant $c$.

We show that condition (4) for $p = 2$ is equivalent to (3) (see [19]). In addition, from (5) it follows that $B(z) \in \Lambda^p_1$ for all $p \in (0, \infty)$ [13].

**Proof.** The implications (2) $\Rightarrow$ (3) $\Rightarrow$ (4) is obvious, (5) $\Rightarrow$ (2) follows from the estimate

$$r^{k-2}k(k-1)|\hat{B}(k)| \leq \frac{1}{2\pi} \int_T |B''(r\zeta)||d\zeta|$$

for $r = 1 - \frac{1}{k}$. We show that (1) $\Rightarrow$ (5) and (4) $\Rightarrow$ (1).

(1) $\Rightarrow$ (5). We make use of the easily proved estimate [4]

$$|B''(z)| \leq 2 \sum \frac{1 - |z_j|^2}{|1 - \bar{z}_j z|^3} + (\sum \frac{1 - |z_j|^2}{|1 - \bar{z}_j z|^2})^2.$$

Let $z = r\zeta$ and integrate with respect to $\zeta \in T$, then we obtain

$$\int_T |B''(r\zeta)||d\zeta| \leq 2 \sum (1 - |z_j|^2) \int_T |1 - \bar{z}_j r\zeta|^{-3} |d\zeta|$$

$$+ \{\sum (1 - |z_j|^2)(\int_T |1 - \bar{z}_j r\zeta|^{-4}|d\zeta|)^{\frac{1}{2}}\}^2.$$
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Since

\[ \int_T |1 - \bar{z}_j r \zeta|^{-n} |d\zeta| \leq c(1 - r|z_j|)^{1-n}, \quad (n > 1), \]

we have

\[ \int_T |B''(r \zeta)||d\zeta| \leq c \sum \frac{1 - |z_j|^2}{(1 - r|z_j|)^2} + c \left( \frac{1 - |z_j|^2}{(1 - r|z_j|)^\frac{3}{2}} \right)^2. \]

From the condition \( \hat{B}(k) = O\left(\frac{1}{k}\right) \) it follows [18] that for \( \alpha > 1 \) one has

\[ \sum (1 - |z_j|^2)(1 - r|z_j|)^{-\alpha} \leq c(1 - r)^{1-\alpha}. \]

Applying this inequality for \( \alpha = 2 \) and \( \alpha = \frac{3}{2} \), we obtain

\[ \int_T |B''(r \zeta)||d\zeta| \leq c(1 - r)^{-1} \]

for some constant \( c \).

(4) \( \Rightarrow \) (1). Let \( B(z) \in \Lambda^p_\alpha \), \( \alpha = \frac{1}{p}, \ p \in (1, \infty) \). From the known results regarding the approximation by Abel means there follows [8] that

\[ \left( \int_T |B(\zeta) - B(r \zeta)|^p|d\zeta| \right)^{\frac{1}{p}} \leq c(1 - r)^\alpha, \]

whence

\[ \int_T (1 - |B(r \zeta)|)^p|d\zeta| \leq c(1 - r)^{\alpha p}. \]

By using the Carleson measure, the proof is complete.

Now we apply the derivative of \( B(z) \) to the deformation of \( B^p \) and find the condition that derivative of Blaschke product belongs to \( A^{p,q} \) spaces. Of course we restrict the value of \( p \) within \( 0 < p < 1 \).

**Theorem 6.** Let \( B(z) \) be a Blaschke product with zeros \( \{a_n\} \) such that \( \sum (1 - |a_n|)^q \) is finite for some \( q \) \( (0 < q < 1) \). Then the condition of \( p \) \( (0 < p < \frac{1}{2q}) \) implies \( B'(z) \in A^{p,q} \).

In order to prove this theorem we use the following lemma.
Lemma 7[17]. Let \{a_n\} be a sequence in \(D\). Then there exists constants \(K, K_p(: dependence on \(p\)) such that

\[
\int_0^{2\pi} \frac{1}{|1 - a_n re^{i\theta}|^{2p}} d\theta \leq \begin{cases} 
\frac{K_p}{(1 - |a_n|^r)^{2p-1}} & \text{if } p > \frac{1}{2} \\
K & \text{if } p < \frac{1}{2}.
\end{cases}
\]

Proof of Theorem 6. The derivative of \(B(z)\) is following formula;

\[
B'(z) = \sum \frac{B_n(z)(1 - |a_n|^2)}{(1 - \bar{a}_n z)^2},
\]

where \(B_n(z) = \frac{B(z)(1 - a_n z)}{1 - \bar{a}_n z}\), and this implies that

\[
|B'(z)| \leq \sum \frac{(1 - |a_n|^2)}{|1 - \bar{a}_n z|^2} \leq 2 \sum \frac{(1 - |a_n|)}{|1 - \bar{a}_n z|^2}.
\]

By the hypothesis, for fixed \(q \ (0 < q < 1)\),

\[
|B'(z)|^q \leq 2^q \sum \frac{(1 - |a_n|)^q}{|1 - \bar{a}_n z|^{2q}}.
\]

Integrate each side and use Lemma 7 for each \(q \ (\frac{1}{2} < q < 1)\), then we obtain that

\[
\int_0^1 \int_0^{2\pi} |B'(re^{i\theta})|^q (1 - r)^{\frac{1}{p} - 2} d\theta dr \leq 2^q K_p \sum (1 - |a_n|)^q \int_0^1 (1 - r)^{\frac{1}{p} - 1 - 2q} dr
\]

is finite for \(0 < p < \frac{1}{2q}\). If \(0 < q < \frac{1}{2}\), we get that

\[
\int_0^{2\pi} |B'(re^{i\theta})|^q d\theta \leq 2^q K \sum (1 - |a_n|)^q.
\]

Thus the proof is complete.

We are now prepared to discuss another conditions to find values of \(p\) and \(q\) or relations of its in \(A^{p,q}\) space using the basic estimate of the inequality \(|1 - a_n re^{i\theta}| \geq (1 - r)\).
THEOREM 8. Let \( \{a_n\} \) be a Blaschke sequence with \( \sum (1 - |a_n|) < \infty \) and \( q < \frac{1}{2p} \), then \( B'(z) \in A^{p,q} \) for each \( q > 1 \).

Proof. We consider the estimate derived from the finite Blaschke product as it is difficult to ensure the convergence of \( \sum \frac{1 - |a_n|}{|1 - a_n r e^{i\theta}|^2} \) for given \( q \). Let

\[
B_m(z) = \prod_{n=1}^{m} \frac{|a_n|}{a_n} \frac{a_n - z}{1 - \bar{a}_n z}
\]

be a finite Blaschke product, then the derivative of \( B_m(z) \) is following;

\[
B'_m(z) = \sum_{n=1}^{m} B_n(z) \frac{1 - |a_n|^2}{(1 - \bar{a}_n z)^2},
\]

where \( B_n(z) = B_m(z) \frac{1 - \bar{a}_n z}{1 - a_n z} \). This implies that

\[
|B'_m(re^{i\theta})|^q \leq 2^q \left( \sum_{n=1}^{m} \frac{d_n}{1 - a_n r e^{i\theta}} \right)^q
\]

for \( 1 - |a_n| = d_n \) \( (n = 1, 2, 3, \ldots) \). By the Hölder inequality, we have

\[
|B'_m(re^{i\theta})|^q \leq 2^q \left( \sum_{n=1}^{m} \left( \frac{1}{d_n} \right)^{\frac{1}{q'}} \right)^{\frac{1}{q'}} \left( \sum_{n=1}^{m} \frac{d_n}{1 - a_n r e^{i\theta}} \right)^q
\]

\[
= 2^q \left( \sum_{n=1}^{m} \frac{d_n}{d_n} \right)^{\frac{1}{q'}} \sum_{n=1}^{m} \frac{d_n}{1 - a_n r e^{i\theta}}^{2q}
\]

where \( \frac{1}{q} + \frac{1}{q'} = 1 \). By Lemma 7,

\[
\int_{0}^{2\pi} |B'_m(re^{i\theta})|^q d\theta \leq 2^q \left( \sum_{n=1}^{m} \frac{d_n}{d_n} \right)^{\frac{1}{q'}} \sum_{n=1}^{m} d_n \int_{0}^{2\pi} \frac{d\theta}{|1 - a_n r e^{i\theta}|^{2q}}
\]

\[
\leq 2^q K_q \left( \sum_{n=1}^{m} d_n \right)^{\frac{4}{q} + 1} (1 - r)^{-2q + 1}.
\]

Since \( \sum d_n \) is finite, the value of the right side of the preceding inequality is finite independently of the choice \( m \). Therefore, we have
the following by the Lebesgue’s theorem,
\[
\int_0^{2\pi} |B'(re^{i\theta})|^q d\theta = \lim_{m \to -\infty} \int_0^{2\pi} |B_m'(re^{i\theta})|^q d\theta \\
\leq 2^q K_q \left( \sum d_n \right)^{\frac{q}{r}+1} (1-r)^{-2q+1}.
\]
Thus
\[
\int_0^1 \int_0^{2\pi} |B'(re^{i\theta})|^q (1-r)^{\frac{1}{r}-2} d\theta dr \\
= \lim_{m \to -\infty} \int_0^1 \int_0^{2\pi} |B_m'(re^{i\theta})|^q (1-r)^{\frac{1}{r}-2} d\theta dr \\
\leq 2^q K_q \left( \sum d_n \right)^{\frac{q}{r}+1} \int_0^1 (1-r)^{\frac{1}{r}-2q+1} dr.
\]
By the hypothesis, this integration is finite for \( q < \frac{1}{2p} \). Therefore the proof is complete.

We notice the convergent relation of \( \sum (1-|a_n|)^q \) and \( \sum (1-|a_n|) \) is depend on \( q \) in the proof of the above theorem.

**Remark.** If \( \sum (1-|a_n|)^q \) is finite then \( \sum (1-|a_n|) \) is also finite but the converse does not hold for each \( q < 1 \). On the other hand, this property is opposite to the mentioned argument for each \( q > 1 \).

**References**

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