ON A GENERALIZED ALMOST KAEHLERIAN FINSLER MANIFOLD

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1. Introduction

Let $M$ be a $2n$-dimensional differential manifold admitting an almost complex structure $f^i_j(x)$ and a Finsler metric $g_{ij}(x, y)$ given by

$$g_{ij}(x, y) = \frac{1}{2} \partial_i \partial_j L^2(x, y),$$

where $\partial_i = \partial/\partial y^i$.

If the fundamental function $L(x, y)$ satisfies the so called Rizza condition, that is,

$$L(x, \phi \theta y) = L(x, y)$$

for any $\theta \in R$, where

$$\phi \phi^i_j = \cos \theta \cdot \delta^i_j + \sin \theta \cdot f^i_j,$$

then $M$ is called an almost Hermitian Finsler manifold or simply a Rizza manifold. The almost Hermitian Finsler structure $(f^i_j(x), g_{ij}(x, y))$ was introduced by G. B. Rizza [5]. Afterword, it was studied by some authors. In [1] M. Fukui has proved that if $g_{ij}(x, y)$ and $f^i_j(x)$ satisfies the condition

$$g_{ij}(x, y) - g_{pq}(x, y)f^p_i(x)f^q_j(x) = 0,$$

then $g_{ij}$ is a Riemannian metric, that is, $(f^i_j, g_{ij})$ is an almost Hermitian structure. In [2] it is known that the Rizza condition (1.2) is equivalent to any one of the following

1. $g_{pq}(x, \phi \theta y)\phi \phi^p_i, \phi \phi^q_j = g_{ij}(x, y),$
2. $g_{ij}(x, y)f^k_i(x)y^ky^j = 0,$
3. $(g_{im}(x, y) - g_{pq}f^p_i(x)f^q_m(x))y^m = 0,$
4. $g_{lm}(x, y)f^m_j(x) + g_{jm}(x, y)f^m_i(x) + 2C_{ijm}(x, y)f^m_r(x)y^r = 0.$

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We define in the present paper a generalized Finsler connection $\tilde{\Gamma}$ and a generalized almost Kaehlerian Finsler manifold with respect to $\tilde{\Gamma}$.

The purpose of the present paper is to study the generalized almost Kaehlerian Finsler manifold.

Throughout the present paper we shall use the terminology and notations in Matsumoto's monograph [3].

2. A Generalized Finsler connection

Let $M$ be an almost Hermitian Finsler manifold with the Rizza structure $(f^i_j(x), g_{ij}(x,y))$. If we put

$$ (2.1) \quad \tilde{g}_{ij} = \frac{1}{2}(g_{ij}(x,y) + g_{pq}(x,y)f^p_i(x)f^q_j(x)),$$

then $\tilde{g}_{ij}$ is a homogeneous symmetric Finsler metric, which is called a generalized Finsler metric induced from the Finsler metric $g_{ij}$. It should easily verified that

$$ \tilde{g}_{ij}(x,y) = \tilde{g}_{pq}(x,y)f^p_i(x)f^q_j(x), $$

from which

$$ (2.2) \quad \tilde{g}_{im}(x,y)f^m_j(x) = -\tilde{g}_{jm}(x,y)f^m_i(x). $$

Concerning the reciprocal tensor $\tilde{g}^{ij}(x,y)$ of $\tilde{g}_{ij}(x,y)$, we could prove

$$ (2.3) \quad \tilde{g}^{ij}(x,y) = \tilde{g}^{kl}(x,y)f^i_k(x)f^j_l(x) $$

from which

$$ \tilde{g}^{ik}(x,y)f^i_k(x) = -\tilde{g}^{jk}(x,y)f^i_k(x). $$

Now we put

$$ (2.4) \quad f_{ij}(x,y) = g_{im}(x,y)f^m_j, $$

and

$$ (2.5) \quad \tilde{f}_{ij}(x,y) = \tilde{g}_{im}(x,y)f^m_j. $$
By virtue of (2.2) we have

\[ \tilde{f}_{ij}(x, y) = -\tilde{f}_{ji}(x, y), \quad \tilde{f}_{im}(x, y)f^m_j(x) = -\tilde{g}_{ij}(x, y), \]

\[ \tilde{f}_{ij}(x, y) = \frac{1}{2}(f_{ij}(x, y) - f_{ji}(x, y)). \]

Let us consider such connection that

\[ \tilde{\Gamma}^i_{jk} = \frac{1}{2}\tilde{g}^{im}(X_k\tilde{g}_{jm} + X_j\tilde{g}_{mk} - X_m\tilde{g}_{kj}), \]

where \( X_k = \partial_k - N^l_k(x, y)\partial_l \), \( N^l_k \) is a non-linear connection and \( \partial_k = \partial/\partial x \). Then \( \tilde{\Gamma}^i_{jk} \) is symmetric and satisfies the transformation rule of a linear connection. So, we represent by \( \tilde{\nabla}_k \) the \( h \)-covariant derivative with respect to \( (\tilde{\Gamma}^i_{jk}, N^l_k) \).

For any Finsler tensor \( T^i_j(x, y) \) of \((1,1)\)-type, the \( h \)-covariant derivative with respect to \( (\tilde{\Gamma}^i_{jk}, N^l_k) \) are expressed as follows:

\[ \tilde{\nabla}_k T^i_j = \partial_k T^i_j - N^l_k T^r_l T^i_j + \tilde{\Gamma}^i_{rk} T^r_j - T^r_i \tilde{\Gamma}^r_{jk}. \]

Therefore, for the almost complex structure tensor \( f^i_j(x) \)

\[ \tilde{\nabla}_k f^i_j(x) = \partial_k f^i_j(x) + \tilde{\Gamma}^i_{rk} f^r_j(x) - f^r_i(x) \tilde{\Gamma}^r_{jk}. \]

Using (2.7) we obtain

\[ \tilde{\nabla}_k \tilde{g}_{ij} = \partial_k \tilde{g}_{ij} - N^l_k \partial_l \tilde{g}_{ij} - X_k \tilde{g}_{ij} = 0. \]

Thus we have

**Theorem 2.1.** A Finsler space with a generalized Hermitian structure \((\tilde{\Gamma}^i_{jk}, N^l_k)\) is \( h \)-metrical.

**3. A generalized almost Kaehlerian Finsler manifold**

A generalized Hermitian Finsler manifold \( M \) with a \((f^i_j(x), \tilde{g}_{ij}(x, y), N)\)-structure satisfying \( \tilde{\nabla} f^i_j = 0 \) is called a generalized Kaehlerian
Finsler manifold [3], and $M$ satisfying $\tilde{\nabla}_k f^i_j + \tilde{\nabla}_j f^i_k = 0$ is said a
generalized nearly Kaehlerian Finsler manifold.

Now, in a generalized Hermitian Finsler manifold $M$ with
$(f^i_j(x), \bar{g}_{ij}(x, y), N)$-structure, we put
$$\tilde{F}_{ijk} + X_i \tilde{f}_{jk} + X_j \tilde{f}_{ki} + X_k \tilde{f}_{ij},$$
then from (2.6) we have

$$(3.1) \quad \tilde{F}_{ijk} = \tilde{\nabla}_i \tilde{f}_{jk} + \tilde{\nabla}_j \tilde{f}_{ki} + \tilde{\nabla}_k \tilde{f}_{ij}.$$ A generalized Hermitian Finsler manifold $M$ with a $(f^i_j(x), \bar{g}_{ij}(x, y), N)$-structure satisfying $\tilde{F}_{ijk} = 0$ is called a generalized almost Kaehlerian Finsler manifold, which following the example of complex Riemannian geometry.

On the other hand the Nijenhuis tensor $N^i_{jk}$ of almost complex structure $f^i_j(x)$ is defined as follos [7]:
$$N^i_{jk} = (\partial_i f^r_j) f^r_k - (\partial_r f^i_j) f^r_k + f^r_i \partial_j f^r_k - f^r_i \partial_k f^r_j.$$ Substituting (2.9) in the above equation we have

$$(3.2) \quad N^i_{jk} = (\tilde{\nabla}_r f^i_j - \tilde{\nabla}_m f^r m_j + f^r_i \tilde{\nabla}_m f^r m_j) f^r_k$$
$$(3.2) \quad + f^r_i (\tilde{\nabla}_j f^r_k - \tilde{\nabla}_m f^r m_k - f^r_m \tilde{\nabla}_k f^r m_j)$$
$$(3.2) \quad - f^r_i (\tilde{\nabla}_k f^r_j - \tilde{\nabla}_m f^r m_j - f^r_m \tilde{\nabla}_j f^r m_k)$$
$$(3.2) \quad = (\tilde{\nabla}_r f^i_j) f^r_k - (\tilde{\nabla}_r f^i_k) f^r_j + f^r_i \tilde{\nabla}_j f^r k - f^r_i \tilde{\nabla}_k f^r j.$$ Moreover let us put $\tilde{N}_{hij} = \tilde{g}_{hm} N^m_{ij}$. Then we have

$$(3.3) \quad \tilde{N}_{hij} = (\tilde{\nabla}_r f^i_j) f^r_k - (\tilde{\nabla}_r f^i_k) f^r_j + \tilde{f}_{hr} \tilde{\nabla}_j f^r i - \tilde{f}_{hr} \tilde{\nabla}_j f^r i$$
by virtue of (2.5) and (2.10).

From (2.6), (3.1) and (3.3) is reduced to

$$(3.4) \quad \tilde{N}_{hij} = f^r_j \tilde{F}_{rh} - f^r_i \tilde{F}_{rh} - 2 \tilde{f}_{jr} \tilde{\nabla}_h f^r i.$$ Since $\tilde{F}_{ijk} = 0$ in a generalized almost Kaehlerian Finsler manifold, we have

$$\tilde{N}_{hij} = -2 \tilde{g}_{jm} f^m r \tilde{\nabla}_h f^r i.$$ Thus we have
Theorem 3.1. A generalized almost Kaehlerian Finsler manifold is a generalized Kaehlerian Finsler manifold if and only if $N_{hj} = 0$.

From (3.4) we have

$$N_{hj} + N_{jkh} = -f^r_1 F^r_{hj} - f^r_h F_{r1j} - 2f_j (\nabla_h f^r_1 + \nabla_i f^r_h).$$

In a generalized almost Kaehlerian Finsler manifold we get

$$N_{h1j} + N_{1kh} = -\tilde{g}_{jm} f^m_r (\nabla_h f^r_1 + \nabla_i f^r_h).$$

Thus we have

Theorem 3.2. A generalized almost Kaehlerian Finsler manifold is a generalized nearly Kaehlerian Finsler manifold if and only if $N_{h1j} + N_{1kh} = 0$.

References

5. H. S. Park, On nearly Kaehlerian Finsler manifolds, Tensor N. S., 52 (1993), 243-248

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