PROPERTIES OF THE CONTINUATION OF COMPLEX ANALYTIC MANIFOLDS SPREAD IN $C^n$

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In the theory of one complex variable, it is well known that holomorphic functions cannot be extended as holomorphic functions to larger domains. In the space $C^n$, a domain of holomorphy has important meanings and is a fruitfull domain in which many function theoretical results are obtained. H. J. Bremmermann [1], F. Norguet [5] and K. Oka [6] solved this problem in $C^n$ and in the unramified domain over $C^n$, F. Docquier and H. Grauert [3] solved this problem in Stein manifold, and J. Kajiwara and K. H. Shon [4] obtained the continuation and vanishing theorem for cohomology of infinite dimensional spaces. We will introduce some properties of the continuation of holomorphic functions and a domain of holomorphy.

1. Preliminary and notation

Let $D$ be a Hausdorff topological space, $\varphi$ be a local homeomorphism on $D$ into $C^n = \{z = (z_1, z_2, \ldots, z_n) \in \mathbb{C}, 1 \leq j \leq n\}$ and $(D, \varphi)$ be a (Riemann) domain over $C^n$, i.e., $\varphi$ spread of $D$ in $C^n$.

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DEFINITION 1.1. A domain over $C^n$ is a pair $(D, \varphi)$ with the following properties:

1) $D$ is a connected topological space.
2) For every two points $x_1, x_2 \in D$ with $x_1 \neq x_2$ there are open neighborhoods $U_1 = U_1(x_1) \subset D, U_2 = U_2(x_2) \subset D$ with $U_1 \cap U_2 = \emptyset$.
3) $\varphi : D \rightarrow C^n$ is a locally homeomorphism.

DEFINITION 1.2. A triple $(\lambda, D', \varphi')$ is an analytic completion of a domain $(D, \varphi)$ over $C^n$ if

1) $(D', \varphi')$ is a domain over $C^n$ i.e. $\varphi' : D' \rightarrow C^n$ is a locally homeomorphism.
2) $\lambda : D \rightarrow D'$ is a local homeomorphism.
3) $\varphi = \varphi' \circ \lambda$.
4) For any holomorphic function $f$ on $D$, there is a holomorphic function $\tilde{f}$ on $D'$ such that $f = \tilde{f} \circ \lambda$.

We say that the above described diagram is commutative if $\varphi = \varphi' \circ \lambda$ and $f = \tilde{f} \circ \lambda$, and we say that $\tilde{f}$ is an analytic prolongation of $f$ to $(\lambda, D', \varphi')$.

In case that $n = 1$, there is no analytic completion. In case of $n \geq 2$, we let $z = (z_1, z_2, \cdots, z_n), z' = (z_2, z_3, \cdots, z_n)$ and $z = (z, z') \in D$ where

$$D = \{ z \in C^n | r^2 < |z_1|^2 + |z_2|^2 + \cdots + |z_n|^2 < R^2 \}.$$ 

If $z$ is a point of $D$ then

$$\sqrt{r^2 - |z_2|^2 - |z_3|^2 - \cdots - |z_n|^2} < |z_1| < \sqrt{R^2 - |z_2|^2 - |z_3|^2 - \cdots - |z_n|^2}.$$ 

For $z' \in C^{n-1}$ with $|z_2|^2 + |z_3|^2 + \cdots + |z_n|^2 > r^2$ we have

$$|z_1|^2 + |z_2|^2 + \cdots + |z_n|^2 > r^2$$

and

$$z \in D \Leftrightarrow |z_1| < \sqrt{R^2 - |z_2|^2 - |z_3|^2 - \cdots - |z_n|^2}.$$
In this case,
\[ \Delta(z') = \{ z_1 \in C | \sqrt{r^2 - |z_2|^2 - |z_3|^2 - \cdots - |z_n|^2} < |z_1| \}
\]
\[ < \sqrt{R^2 - |z_2|^2 - |z_3|^2 - \cdots - |z_n|^2} \}
\[ = \{ z_1 \in C | |z_1| < \sqrt{R^2 - |z_2|^2 - |z_3|^2 - \cdots - |z_n|^2} \}. \]

For \( z' \in C_{n-1} \) with \( |z_2|^2 + |z_3|^2 + \cdots + |z_n|^2 \leq r^2 \), \( \Delta(z') \) is the of form
\[ \{ z \in C | 0 < |z_1| < \sqrt{R^2 - |z_2|^2 - |z_3|^2 - \cdots - |z_n|^2} \}
\] or
\[ \{ z \in C | \sqrt{r^2 - |z_2|^2 - |z_3|^2 - \cdots - |z_n|^2} < |z_1| \}
\[ < \sqrt{R^2 - |z_2|^2 - |z_3|^2 - \cdots - |z_n|^2} \}. \]

Thus we obtain a well defined holomorphic function \( f' \) on \( D' = \{ z \in C^n | |z_1|^2 + |z_2|^2 + \cdots + |z_n|^2 < R^2 \} \). Therefore, a holomorphic function of \( n \geq 2 \) complex variables cannot have isolated singularities. Furthermore, it cannot have isolated zeros.

**Definition 1.3.** Let \( (D, \varphi) \) be a domain over a (connected) complex analytic manifold \( M \) and \( \mathcal{H}(D) \) be a family of holomorphic functions on \( D \). A triple \( (\tilde{\lambda}, \tilde{D}, \tilde{\varphi}) = (\hat{D}, \hat{\varphi}, \hat{\lambda}, \hat{f}) \) is an envelope of holomorphy (or a continuation) of \( (D, \varphi) = (D, \varphi, f) \) with respect to \( \mathcal{H}(D) \) if

1) \( (\tilde{\lambda}, \tilde{D}, \tilde{\varphi}) \) is an analytic completion of \( (D, \varphi) \) with respect to \( \mathcal{H}(D) \).

2) For any analytic completion \( (\hat{\lambda}, \hat{D}, \hat{\varphi}) \) of \( (D, \varphi) \), there is a local homeomorphism \( \mu : D' \rightarrow \hat{D} \) such that \( (\mu, \tilde{\lambda}, \tilde{\varphi}) \) is an analytic completion of \( (D', \varphi') \) with respect to the family of all analytic continuations of functions of \( \mathcal{H}(D) \).

Let \( V \) be a complex (analytic) manifold and \( \mathcal{H}(V) \) be a family of holomorphic functions on \( V \). A triple \( (\lambda, V', \varphi') \) is an envelope of holomorphy of \( (V, \varphi) \) with respect to \( \mathcal{H}(V) \) if

1) \( (V', \varphi') \) is a domain over \( C^n \).

2) For any \( f \) in \( \mathcal{H}(V) \), there is a holomorphic function \( f' \) on \( V' \) such that \( f = f' \circ \lambda \) and \( \varphi = \varphi' \circ \lambda \).

3) \( \lambda : V \rightarrow V' \) is a local homeomorphism.
2. Geometrically properties of complex domains

H. Cartan [2] proved the unique existence of such envelope of holomorphy. Especially if \(\mathcal{H}(D)\) consists of only one holomorphic function \(f\) on \(D\), the envelope of holomorphy of \((D, \varphi)\) with respect to \(\mathcal{H}(D)\) is called a domain of holomorphy of \(f\). A domain over \(M\) which is a domain of holomorphy of a holomorphic function on a domain over \(M\) is called shortly a domain of holomorphy. Moreover if \(\mathcal{H}(D)\) is the family of all holomorphic functions on \(D\), the envelope of holomorphy of \((D, \varphi)\) with respect to \(\mathcal{H}(D)\) is called shortly the envelope of holomorphy of \((D, \varphi)\).

**Definition 2.1.** The maximal continuation of \((D, \varphi, \mathcal{H}(D))\) is called the envelope of holomorphy of \((D, \varphi)\).

**Theorem 2.2.** Let \(V\) be a complex manifold, \((V, \varphi_j)(j = 1, 2)\) be two domains over \(C^n\), and let \((V_j, \mu, F_j, \mathcal{H}(V_j))\) be the maximal continuation of \((V, \varphi_j, \mathcal{H}(V))\) for \(j = 1, 2\). Then \(V_1, V_2\) are homeomorphic under a mapping \(G : V_2 \rightarrow V_1\) such that the following diagram

\[
\begin{array}{ccc}
C^n & \xleftarrow{\varphi_1} & V & \xrightarrow{\varphi_2} & C^n \\
\mu_1 \uparrow & & \uparrow \mu_2 & & \\
V_1 & \leftarrow G & V_2
\end{array}
\]

with local homeomorphisms \(F_1 : V \rightarrow V_1\) and \(F_2 : V \rightarrow V_2\) is full commutative.

**Proof.** If \(\varphi = (\varphi_{11}, \varphi_{12}, \ldots, \varphi_{1n})\), then the functions \(\varphi_{1j} \in \mathcal{H}(V)\) and so can be continued to functions \(\varphi^*_1 = (\varphi^*_{11}, \varphi^*_{12}, \ldots, \varphi^*_{1n})\) on \(V_2\). Since \(\varphi_1, \varphi_2\) are local homeomorphic, the Jacobian \(J = \frac{\partial \varphi_2}{\partial \varphi_1}\) never vanishes on \(V\) and \(J \in \mathcal{H}(V)\). Both \(J\) and \(J^{-1}\) can be continued to functions \(J^*\) and \((J^{-1})^*\) in \(\mathcal{H}(V_2)\). By the principle of analytic continuation we have \(J \circ J^{-1} \equiv 1\) on \(V\). Hence \(J^*(J^{-1})^* \equiv 1\) on \(V_2\). Therefore \(J^*\) never vanishes. And \(J^* = \frac{\partial \varphi^*_1}{\partial \mu_2}\), so that \(\varphi^*_1\) is a local homeomorphism and \((V_2, \varphi^*_1)\) is a domain over \(C^n\). Since \(\varphi_1 = \varphi^*_1 \circ F_2\), \((V_2, \varphi^*_1, F_2, \mathcal{H}(V_2))\) is a continuation of \((V, \varphi, \mathcal{H}(M))\), and maximality
of \((V_1, \mu_1, F_1, \mathcal{H}(V_1))\) implies the existence of a domain \((V_2, G)\) over \(V_1\) such that the following diagram commutes.

\[
\begin{array}{ccc}
V & \overset{F_1}{\longrightarrow} & V_1 \\
& \searrow & \uparrow G \\
& F_2 & \varphi_1
\end{array}
\]

\(V_2\)

Similarly extending \(\varphi_2\) to \(\varphi_2^*\) we find \(\varphi_2^*\) is a local homeomorphic spreading \(V_1\) in \(C^n\) such that \(\varphi_2 = \varphi_2^* \circ F_1\). If we use maximality of \((V_2, \mu_2, F_2, \mathcal{H}(V_2))\), we obtain a mapping \(\tilde{G}\) which spread \(V_1\) in \(V_2\) and gives the following commutative diagram.

\[
\begin{array}{ccc}
V & \overset{F_2}{\longrightarrow} & V_2 \\
& \searrow & \uparrow \tilde{G} \\
& F_1 & \varphi_2^*
\end{array}
\]

\(V_1\)

Hence the above diagram with \(\varphi_2 : V \longrightarrow C^n\) is commutative. If \(a \in V\) and \(U\) is a small neighborhood of \(a\), then \(F_2\) is homeomorphic on \(U\) into \(V_2\). On \(F_2(U)\), we have

\[
\tilde{G}G(y) = y \iff \tilde{G}G(F_2(x)) = F_2(x)
\]

for all \(x \in U\). But from the above diagrams, we know that

\[
F_2^{-1} \tilde{G}GF_2(x) = F_2^{-1} \tilde{G}G_1(x) = F_2^{-1} F_2(x) = x.
\]

Therefore, from the analytic continuation, \(\tilde{G}G(y) = y\) on \(V_2\). Similarly, \(G \tilde{G}(x) = x\) on \(V_1\) and \(G\) is global homeomorphic.

**Corollary 2.3.** The envelope of holomorphy is not univalent.

**Proposition 2.4.** Let \(M = \{(z_1, z_2) \in C^2 | |z_2| < Re(z_1) < |z_2| + 6\}\) and

\[
\varphi : (z_1, z_2) \rightarrow (e^{iz_1}, z_2).
\]
If $\tilde{M} = \varphi(M)$, then $\varphi$ is a homeomorphism of $M$ onto $\tilde{M}$. Any $f \in \mathcal{H}(M)$ can be extended to $N = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_2| < \text{Re}(z_1)\}$.

Proof. We know the first part from $6 < 2\pi$. If $\text{Re}(z_1) > 0$, then $f(z, \zeta)$ is defined and holomorphic for all $\zeta$ with $\text{Re}(z_1) - 6 < |\zeta| < \text{Re}(z_1)$, and we see that $f$ extends by examining the coefficient functions $\varphi_k(z_1)(k < 0)$ in the Laurent expansion $f(z_1, \zeta) = \sum_{k=-\infty}^{\infty} \varphi_k(z_1) \zeta^k$.

But $N$ is a domain of holomorphy because it is convex, and this means that $(N, \varphi, \tilde{\mathcal{H}}, i_1)$ is the envelope of holomorphy of $(M, \varphi, \mathcal{H}(M))$ where $i_1 = \text{id}_M$ and

$$\tilde{\mathcal{H}} = \{\tilde{f} \in \mathcal{H}(N) \mid \tilde{f} \text{extends } f \text{ to } N, f \in \mathcal{H}(M)\}$$

and $\varphi$ is identified with the natural spread of the envelope of holomorphy in $\mathbb{C}^n$.

Remark 2.5. If $U \subset M$ is a compact set, and $\tilde{U} = i_1(U)$ its images in $N$ then $\varphi^{-1}(\varphi(\tilde{U}))$ is usually not compact.

References


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