Bull. Korean Math. Soc. **32** (1995), No. 1, pp. 19–23

THE EIGENVALUE ESTIMATE ON A COMPACT RIEMANNIAN MANIFOLD

BANG OK KIM AND KWON WOOK KIM

1. Introduction

We will estimate the lower bound of the first nonzero Neumann and Dirichlet eigenvalue of Laplacian equation on compact Riemannian manifold M with boundary. In case that the boundary of M has positive second fundamental form elements, Li-Yau[3] gave the lower bound of the first nonzero neumann eigenvalue η_1. In case that the second fundamental form elements of ∂M is bounded below by negative constant, Roger Chen[4] investigated the lower bound of η_1. In [1], [2], we obtained the lower bound of the first nonzero Neumann eigenvalue η_1 and Dirichlet eigenvalue λ_1 under the condition of [4]. In this paper, the lower bound of the first nonzero Neumann eigenvalue is estimated under the condition that the second fundamental form elements of boundary is bounded below by zero. Moreover, I realize that " the interior rolling ε – ball condition " is not necessary when the first Dirichlet eigenvalue was estimated in [1].

THEOREM 1. *Let M be an n-dimensional compact Riemannian manifold with boundary ∂M. Let R and A be positive constants such that the Ricci curvature of M is bounded below by $-R$, the sectional curvature of M is bounded above by A, the second fundamental form elements of ∂M is bounded below by zero.*

If u is a solution of the equation

$$\Delta u + \eta_1 u = 0 \text{ in } M$$

$$\frac{\partial u}{\partial \nu} \equiv 0 \text{ on } \partial M,$$

Received June 14, 1993.
This research was supported by TGRC-KOSEF.

where ν is the unit outward normal vector to ∂M. Then

$$\eta_1 \geq \frac{1+\sqrt{1+2(n-1)d^2R}}{(n-1)d^2}\exp(-(1+\sqrt{1+2(n-1)d^2R})),$$

where d is the diameter of M.

Proof. Let $\psi_m(r)$ be a nonnegative C^2-function defined on $[0,\infty)$ such that, for $\varepsilon < \frac{\pi}{2\sqrt{A}}$ and any positive integer m,

$$\psi_m(r) = \begin{cases} -\frac{1}{2\varepsilon m}(r-\varepsilon)^2 + \frac{\varepsilon}{2m} & \text{if} \quad r \in [0,\varepsilon) \\ \frac{\varepsilon}{2m} & \text{if} \quad r \in [\varepsilon,\infty) \end{cases}$$

Define $\phi_m(x) = \psi_m(r(x))$, where $r(x)$ denotes the distance function from boundary ∂M to $x \in M$. For $\beta > \sup u$, we define the auxiliary function

$$G_m(x) = (1+\phi_m)^{\frac{1}{2}}\frac{|\nabla u|^2}{(\beta-u)^2} \quad \text{on} \quad M.$$

By the compactness of M, there is a point $x_0 \in M$ such that $G_m(x)$ achives its supremum. Suppose that x_0 is a boundary point of ∂M. At x_0, we may choose an orthonormal frame field $e_1, e_2, \ldots, e_n$ such that $e_n = \frac{\partial}{\partial \nu}$, where $\frac{\partial}{\partial \nu}$ is the unit outward normal vector. Then we have

$$0 \leq \frac{\partial G_m}{\partial \nu}(x_0) = -\frac{1}{2}\frac{|\nabla u|^2}{(\beta-u)^2}\frac{1}{m} \quad < 0,$$

which is a contradiction. Therefore x_0 has to be an interior point of M.

From the fact that $\Delta G_m(x_0) \leq 0$ and $\nabla G_m(x_0) = 0$, we obtain that
(1)

$$0 \geq \left(\frac{1-\alpha^2}{n-1}\right)G_m(x_0)^2 - \left\{\frac{(2n-3)^2+\alpha^2(10n-11)}{16\alpha^2(n-1)}(1+\phi_m)^{-\frac{3}{2}}|\nabla\phi_m|^2 \right.$$
$$\left. +2(1+\phi_m)^{\frac{1}{2}}(R+\eta_1) + \frac{2\eta_1 u}{\beta-u}(1+\phi_m)^{\frac{1}{2}} - \frac{1}{2}\,\Delta\,\phi_m(1+\phi_m)^{-\frac{1}{2}}\right\}G_m(x_0)$$
$$-\frac{2(1+\phi_m)\eta_1^2u^2}{(\beta-u)^2(n-1)}, \qquad \text{for} \quad 0 < \alpha \leq \frac{1}{\sqrt{3}}.$$

If $r(x_0) \geq \varepsilon$, then $\Delta\phi_m(x_0) = 0$.

If $r(x_0) < \varepsilon$, then $\Delta r(x_0) \geq (-1)(n-1)\sqrt{A}\tan\sqrt{A}\varepsilon$.
Hence we have

$$\Delta\phi_m(x_0) \geq -\frac{1}{\varepsilon m} - (n-1)\frac{\sqrt{A}}{m}\tan\sqrt{A}\varepsilon. \tag{2}$$

Substituting (2) to (1), we obtain that

$$\begin{aligned} 0 \geq & \frac{(1-\alpha^2)}{(n-1)}G_m(x_0)^2 - \left\{C + \frac{2\beta}{\beta - \sup u}\left(1+\frac{\varepsilon}{2m}\right)^{\frac{1}{2}}\eta_1\right\}G_m(x_0) \\ & - \frac{2(1+\frac{\varepsilon}{2m})\eta_1^2 u^2}{(\beta - \sup u)^2(n-1)}, \end{aligned} \tag{3}$$

where

$$\begin{aligned} C = & \frac{(2n-3)^2 + \alpha^2(10n-11)}{16\alpha^2(n-1)}\frac{1}{m^2} \\ & + 2\sqrt{1+\frac{\varepsilon}{2m}}R + \frac{n-1}{2m}\sqrt{A}\tan\sqrt{A}\varepsilon + \frac{1}{2\varepsilon m}. \end{aligned}$$

We can assume that $\sup u = 1$ and $\inf u = -k \geq -1$. From (3), we obtain that

$$\eta_1 \geq \frac{1}{\sqrt{1+\frac{\varepsilon}{2m}}}\frac{1-\alpha^2}{(n-1)d^2}(1+B)\exp(-(1+B)),$$

where

$$B = \sqrt{1 + \frac{(n-1)d^2}{1-\alpha^2}C}.$$

Let

$$a_m = \frac{1}{\sqrt{1+\frac{\varepsilon}{2m}}}\frac{1-\alpha^2}{(n-1)d^2}(1+B)\exp(-(1+B)).$$

Since $\{a_m\}$ is an increasing sequence and $\eta_1 \geq a_m$ for all positive integer m, we have that

$$\begin{aligned} \eta_1 \geq & \frac{1-\alpha^2}{(n-1)d^2}\left(1+\sqrt{1+2\frac{(n-1)d^2}{1-\alpha^2}R}\right) \\ & \exp\left(-\left(1+\sqrt{1+2\frac{(n-1)d^2}{1-\alpha^2}R}\right)\right). \end{aligned}$$

Let $f(\alpha)$ be the function defined on $[0, \frac{1}{\sqrt{3}}]$ by

$$\frac{1-\alpha^2}{(n-1)d^2}\Big(1+\sqrt{1+2\frac{(n-1)d^2}{1-\alpha^2}R}\Big)\exp\left(-\Big(1+\sqrt{1+2\frac{(n-1)d^2}{1-\alpha^2}R}\Big)\right).$$

Then $f(\alpha)$ is a continuous and decreasing function on $[0, \frac{1}{\sqrt{3}}]$.

Hence $D = \{f(\alpha) | 0 < \alpha \leq \frac{1}{\sqrt{3}}\}$ has the least upper bound

$$f(0) = \frac{1}{(n-1)d^2}\left(1+\sqrt{1+2(n-1)d^2R}\right)$$
$$\exp\left(-(1+\sqrt{1+2(n-1)d^2R})\right).$$

It follows that $\eta_1 \geq f(0)$.

THEOREM 2. *Let M be an n-dimensional compact Riemannian manifold with boundary ∂M. Let R, K, A and H be positive constants such that the Ricci curvature of M is bounded below by $-R$, the sectional curvature of M is bounded above by A, the mean curvature of ∂M is bounded below by $-K$ and and the second fundamental form elements of ∂M is bounded below by $-H$. If u is a solution of the equation*

$$\Delta u + \lambda_1 u = 0 \text{ in } M \tag{1.2}$$
$$u \equiv 0 \text{ on } \partial M,$$

where λ_1 is the first Dirichlet eigenvalue. Then

$$\lambda_1 \geq \frac{1}{\sqrt{1+2(n-1)\varepsilon K}}\,\frac{(1-\alpha^2)}{(n-1)\rho^2}(1+B)\exp(-(1+B)),$$

where

$$B = \left\{1+\frac{(n-1)\rho^2}{1-\alpha^2}C\right\}^{\frac{1}{2}},$$
$$C = \frac{(2n-3)^2+\alpha^2(10n-11)}{\alpha^2}(n-1)K^2 + 2(1+\varepsilon K)^{\frac{1}{2}}R$$
$$+4(n-1)K\left(\frac{1}{\varepsilon}+(n-1)\frac{H+\sqrt{A}\tan(\varepsilon\sqrt{A})}{1-\frac{H}{\sqrt{A}}\tan(\varepsilon\sqrt{A})}\right),$$
$$0<\alpha\leq\frac{1}{\sqrt{3}}, \quad \frac{H}{\sqrt{A}}\tan\sqrt{A}\varepsilon < 1.$$

and ρ is the radius of the largest geodesic ball contained in M.

Proof. Let $\psi(r)$ be a nonnegative C^2-function defined on $[0,\infty)$ such that, for $\frac{H}{\sqrt{A}}\tan\sqrt{A}\varepsilon < 1$,

$$\psi(r) = \begin{cases} -\frac{2}{\varepsilon}(n-1)K(r-\varepsilon)^2 + 2(n-1)\varepsilon K & \text{if} \quad r \in [0,\varepsilon) \\ 2(n-1)\varepsilon K & \text{if} \quad r \in [\varepsilon,\infty) \end{cases}$$

Define $\phi(x) = \psi(r(x))$, where $r(x)$ denotes the distance function from boundary ∂M to $x \in M$. For $\beta > \sup u$, we define the function

$$G(x) = (1+\phi)^{\frac{1}{2}} \frac{|\nabla u|^2}{(\beta-u)^2} \quad \text{on} \quad M.$$

Then G has a maximum at an interior point of M. Using the same method of [1] or theorem 1, we obtain the result.

REMARK. By [5], If $\frac{H}{\sqrt{A}}\tan(\varepsilon\sqrt{A}) < 1$, we can choose a geodesic from boundary to $x(r(x) \le \varepsilon)$ which has no focal point. Hence the interior rolling condition in [1] is not necessary.

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BANG OK KIM
DEPARTMENT OF INDUSTRIAL SAFETY, SUNCHON JUNIOR TECHNICAL COLLEGE, SUNCHON 540-744, KOREA

KWON WOOK KIM
DEPARTMENT OF MATHEMATICS EDUCATIONAL, SUNCHON UNIVERSITY, SUNCHON 540-742, KOREA