

ON ηK -CONFORMAL KILLING TENSOR IN COSYMPLECTIC MANIFOLD WITH VANISHING COSYMPLECTIC BOCHNER CURVATURE TENSOR*

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1. Introduction

S. Tachibana [10] has defined a *conformal Killing tensor* in a n -dimensional Riemannian manifold M by a skew symmetric tensor u_{ji} satisfying the equation

$$\nabla_k u_{ji} + \nabla_j u_{ki} = 2\rho_i g_{kj} - \rho_j g_{ki} - \rho_k g_{ji},$$

where g_{ji} is the metric tensor of M , ∇ denotes the covariant derivative with respect to g_{ji} and ρ_i is a associated covector field of u_{ji} . In here, a covector field means a 1-form.

Also, in a Sasakian manifold M with structure tensor ϕ_j^h , structure vector ξ^h and 1-form η_j , the following equation is satisfied:

$$\nabla_k \phi_{ji} + \nabla_j \phi_{ki} = -(2\eta_i \gamma_{kj} - \eta_j \gamma_{ki} - \eta_k \gamma_{ji}),$$

where we put $\gamma_{ji} = g_{ji} - \eta_j \eta_i$, $\eta_j = g_{jh} \xi^h$ and g_{ji} being the metric tensor of M .

In relation to this fact, Eum [3] has defined an η -conformal Killing tensor by a skew symmetric tensor field u_{ji} satisfying

$$\nabla_k u_{ji} + \nabla_j u_{ki} = 2\rho_i \gamma_{kj} - \rho_j \gamma_{ki} - \rho_k \gamma_{ji}$$

in normal almost contact manifolds.

On the other hand, S. Yamaguchi [12] has introduced a K -conformal Killing tensor in a Kaehlerian manifold and obtained analogous results to a conformal Killing tensor ([11],[12]).

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Recently, in almost contact metric manifolds, one of the authors [4] has introduced a ηK -conformal Killing tensor u_{ji} satisfying the condition

$$(*) \quad \nabla_k u_{ji} + \nabla_j u_{ki} = 2\rho_i \gamma_{kj} - \rho_j \gamma_{ki} - \rho_k \gamma_{ji} + 3(\tilde{\rho}_k \phi_{ji} + \tilde{\rho}_j \phi_{ki}),$$

where $\tilde{\rho}_k = \phi_k^a \rho_a$ and proved the following.

THEOREM [4]. *If there exists a horizontal tensor field u_{ji} whose horizontal associated vector field ρ_j satisfying the condition (*) which takes any preassigned skew symmetric value at any point of a $n(> 3)$ -dimensional cosymplectic manifold M , then M has a vanishing cosymplectic Bochner curvature tensor.*

We call ρ_j in the above theorem, the associated vector of u_{ji} , and if ρ_j vanishes identically, then u_{ji} becomes a Killing one.

The purpose of this paper is to find the existence of a ηK -conformal Killing tensor in a cosymplectic manifold and investigate some geometric properties of the manifold with ηK -conformal Killing tensor.

2. Cosymplectic manifolds

Let M be an $(2n+1)$ -dimensional differentiable manifold of class C^∞ covered by a system of coordinate neighborhoods $\{U : x^h\}$ in which there are given a tensor field ϕ_i^h of type $(1,1)$, a vector field ξ^h and 1-form η_i satisfying

$$(2.1) \quad \phi_j^i \phi_i^h = -\delta_j^h + \eta_j \xi^h, \quad \phi_j^h \xi^j = 0, \quad \eta_i \phi_j^i = 0, \quad \eta_i \xi^i = 1,$$

where, here and in the sequel, the indices h, i, j, \dots run over the range $\{1, 2, 3, \dots, 2n+1\}$. Such a set of a tensor field ϕ of type $(1,1)$, a vector field ξ and 1-form η is called an *almost contact structure* and a manifold with such a structure an *almost contact manifold*.

If, in an almost contact manifold, there is given a positive definite Riemannian metric tensor g_{ji} satisfying

$$(2.2) \quad g_{st} \phi_j^s \phi_i^t = \gamma_{ji}, \quad \eta_i = g_{ih} \xi^h,$$

where $\gamma_j^h = g^{ht}\gamma_{jt}$, then the almost contact manifold is called an *almost contact metric manifold*. By virtue of the last equation of (2.2), we shall write η^h in stead of ξ^h in the sequel. An almost contact metric manifold M with an almost contact metric structure $(\phi_j^h, \eta^h, \eta_j, g_{ji})$ is called an *almost cosymplectic manifold* if 2-form $\Phi = \phi_{ji}dx^j \wedge dx^i$ and 1-form $\omega = \eta_j dx^j$ are both closed, that is, $d\Phi = 0$ and $d\omega = 0$.

If the structure $(\phi_j^h, \eta^h, \eta_j)$ satisfies $N_{ji}^h + (\partial_j \eta_i - \partial_i \eta_j)\eta^h = 0$, where N_{ji}^h is the Nijenhuis tensor formed with ϕ_j^h , then the structure is said to be *normal*.

A normal almost cosymplectic manifold is said to be a *cosymplectic manifold*. It is well known that the cosymplectic structure is characterized by the following ([1]):

$$\nabla_j \phi_{ki} = 0 \quad \text{and} \quad \nabla_j \eta_i = 0.$$

We denote by R_{kji}^h the curvature tensor in a cosymplectic manifold M . It is well known that R_{kji}^h and $R_{kjih} = R_{kji}^a g_{ah}$ satisfy

$$(2.4) \quad \nabla_a R_{kji}^a = \nabla_k R_{ji} - \nabla_j R_{ki},$$

$$(2.5) \quad 2\nabla_a R_k^a = \nabla_k R,$$

where R_{ji} and R are the Ricci tensor and the scalar curvature of M respectively.

By the Ricci identities and (2.3), the following identities are well known :

$$(2.6) \quad R_{kji}^a \eta_a = 0, \quad R_j^a \eta_a = 0,$$

$$(2.7) \quad R_{kja}^h \phi_i^a - R_{kji}^a \phi_a^h = 0, \quad R_{kji}^h + R_{kjs}^a \phi_i^s \phi_a^h = 0,$$

$$(2.8) \quad R_{kjih} - R_{kjsa} \phi_i^s \phi_h^a = 0,$$

$$(2.9) \quad R_j^a \phi_a^h - R_a^h \phi_j^a = 0, \quad R_{ja} \phi_i^a + R_{ia} \phi_j^a = 0.$$

We define H_i^h by

$$(2.10) \quad 2H_i^h = -R_{kji}^h \phi^{kj},$$

where $\phi^{kj} = g^{ka}\phi_a^j$. Then $H_{ih} = H_i^a g_{ah}$ is given by

$$(2.11) \quad 2H_{ih} = -R_{asih}\phi^{as} = -R_{ihas}\phi^{as}.$$

Taking account of (2.6) and (2.11), we can find

$$(2.12) \quad H_{ij} = -H_{ji}, \quad H_{ia}\eta^a = 0.$$

From the first Bianchi's identity and (2.11), we obtain

$$(2.13) \quad H_{ih} = R_{aihs}\phi^{as}.$$

We have from (2.6), (2.8) and (2.13)

$$(2.14) \quad H_{ji} = \phi_j^a R_{ai}, \quad R_{ji} = H_{ja}\phi_i^a,$$

and from which

$$(2.15) \quad R_j^a H_{ak} + R_k^a H_{aj} = 0.$$

Taking account of the second Bianchi's identity, we get, from (2.11) that

$$(2.16) \quad \nabla_j H_{ih} + \nabla_i H_{hj} + \nabla_h H_{ji} = 0,$$

and from (2.4) and (2.5),

$$(2.17) \quad 2\nabla_a H_i^a = (\nabla_a R)\phi_i^a.$$

Differentiating covariantly the second equation of (2.14) and making use of (2.6) and (2.9), we find

$$(2.18) \quad \phi_k^a (\nabla_j H_{ia} - \nabla_i H_{ja}) = \nabla_j R_{ik} - \nabla_i R_{jk},$$

and from (2.12), (2.16) and (2.18),

$$(2.19) \quad \phi_k^a \nabla_a H_{ji} = \nabla_i R_{jk} - \nabla_j R_{ik}.$$

If we denote by $\theta(\eta)$ the Lie derivation with respect to the vector field η^h , we have formally $\theta(\eta)g_{ji} = 0$ with the help of (2.6). From this result, we get using formulae on Lie derivations, $\theta(\eta)R_{kji}{}^h = 0$, $\theta(\eta)R_{ji} = 0$ and $\theta(\eta)R = 0$. Therefore we have

$$(2.20) \quad \eta^a \nabla_a R_{kji}{}^h = 0,$$

$$(2.21) \quad \eta^a \nabla_a R_{ji} = 0,$$

$$(2.22) \quad \eta^a \nabla_a R = 0.$$

Taking account of (2.12), (2.14), and (2.21), we obtain

$$(2.23) \quad \eta^a \nabla_a H_{kj} = 0.$$

3. Cosymplectic Bochner curvature tensor

In this section, we introduce the so-called *cosymplectic Bochner curvature tensor* ([2]) defined by

$$(3.1) \quad B_{kji}{}^h = R_{kji}{}^h + \gamma_k^h L_{ji} - \gamma_j^h L_{ki} + L_k^h \gamma_{ji} - L_j^h \gamma_{ki} + \phi_k^h M_{ji} \\ + M_k^h \phi_{ji} - \phi_j^h M_{ki} - M_j^h \phi_{ki} - 2(M_{kj} \phi_i^h + \phi_{kj} M_i^h),$$

where we put

$$(3.2) \quad L_{ji} = -\frac{1}{2(n+2)}[R_{ji} - \frac{R}{4(n+1)}\gamma_{ji}], \quad L_j^h = g^{ha} L_{ja}, \text{ and}$$

$$(3.3) \quad M_{ji} = -L_{ja} \phi_i^a, \quad M_j^a = g^{ha} M_{jh}.$$

From (2.21), (2.22) and (3.2), we have

$$(3.4) \quad \eta^a \nabla_a L_{ji} = 0.$$

From (2.5) and (3.2), we can easily obtain

$$(3.5) \quad \nabla_a L_k^a = -\frac{2n+1}{8(n+1)(n+2)} \nabla_k R.$$

Substituting (2.14) into (3.3), we find

$$(3.6) \quad M_{ji} = -\frac{1}{2(n+2)} H_{ji} + \frac{R}{8(n+1)(n+2)} \phi_{ji},$$

and from which

$$(3.7) \quad \nabla_k M_{ji} - \nabla_j M_{ki} = -\frac{1}{2(n+2)} (\nabla_k H_{ji} - \nabla_j H_{ki}) \\ + \frac{1}{8(n+1)(n+2)} (\phi_{ji} \nabla_k R - \phi_{ki} \nabla_j R).$$

Transvecting (3.7) with g^{ki} and making use of the skew-symmetries of M_{ji} and H_{ji} , we obtain

$$(3.8) \quad \nabla_a M_j^a = -\frac{1}{2(n+2)} \nabla_a H_j^a + \frac{1}{8(n+1)(n+2)} \phi_j^a \nabla_a R.$$

By virtue of (2.17), the equation (3.8) is reduced to

$$(3.9) \quad \nabla_a M_j^a = -\frac{2n+1}{8(n+1)(n+2)} \phi_j^a \nabla_a R$$

Furthermore, we have from (2.3) and (3.6)

$$(3.10) \quad \begin{aligned} \phi_k^a \nabla_a M_{ji} &= -\frac{1}{2(n+1)} \phi_k^a \nabla_a H_{ji} \\ &\quad + \frac{1}{8(n+1)(n+2)} \phi_k^a \phi_{ji} \nabla_a R. \end{aligned}$$

Substituting (2.19) into (3.10), we obtain

$$(3.11) \quad \begin{aligned} \phi_k^a \nabla_a M_{ji} &= \frac{1}{2(n+1)} (\nabla_j R_{ik} - \nabla_i R_{jk}) \\ &\quad + \frac{1}{8(n+1)(n+2)} \phi_k^a \phi_{ji} \nabla_a R. \end{aligned}$$

On the other hand, from (2.3) and (3.1), we have

$$(3.12) \quad \begin{aligned} \nabla_a B_{kji}^a &= \nabla_a R_{kji}^a + \gamma_k^a \nabla_a L_{ji} - \gamma_j^a \nabla_a L_{ki} + (\nabla_a L_k^a) \gamma_{ji} \\ &\quad - (\nabla_a L_j^a) \gamma_{ki} + \phi_k^a \nabla_a M_{ji} - \phi_j^a \nabla_a M_{ki} \\ &\quad + (\nabla_a M_k^a) \phi_{ji} - (\nabla_a M_j^a) \phi_{ki} \\ &\quad - 2[(\nabla_a M_{kj}) \phi_i^a + \phi_{kj} (\nabla_a M_i^a)]. \end{aligned}$$

Taking account of (2.4), (3.4), (3.5), (3.9) and (3.11), the above equation (3.12) is reduced to

$$(3.13) \quad \begin{aligned} \nabla_a B_{kji}^a &= -2n[\nabla_k L_{ji} - \nabla_j L_{ki} + \frac{1}{8(n+1)(n+2)} \\ &\quad (\phi_k^a \phi_{ji} - \phi_j^a \phi_{ki} - 2\phi_i^a \phi_{kj}) \nabla_a R], \end{aligned}$$

or equivalently

$$(3.14) \quad \nabla_a B_{kji}^a = \frac{n}{n+2} B_{kji},$$

where we put

$$(3.15) \quad \begin{aligned} B_{kji} &= \nabla_k R_{ji} - \nabla_j R_{ki} - \frac{1}{4(n+1)} (\gamma_k^a \gamma_{ji} - \gamma_j^a \gamma_{ki} \\ &\quad + \phi_k^a \phi_{ji} - \phi_j^a \phi_{ki} - 2\phi_i^a \phi_{kj}) \nabla_a R. \end{aligned}$$

4. Main theorem

In this section, we assume that M is a $(2n + 1)$ -dimensional cosymplectic manifold with vanishing cosymplectic Bochner curvature tensor.

First of all, we have from (3.14) that $B_{kji} = 0$. Thus we obtain, from (3.15)

$$(4.1) \quad \nabla_k R_{ji} - \nabla_j R_{ki} = \frac{1}{4(n+1)} (\gamma_k^a \gamma_{ji} - \gamma_j^a \gamma_{ki} + \phi_k^a \phi_{ji} - \phi_j^a \phi_{ki} - 2\phi_i^a \phi_{kj}) \nabla_a R.$$

Transvecting (4.1) with ϕ_s^i and making use of (2.19) and (2.23), we can find

$$(4.2) \quad \nabla_s H_{jk} = -\frac{1}{4(n+1)} (\gamma_k^a \phi_{sj} - \gamma_j^c \phi_{sk} + \phi_k^a \gamma_{sj} - \phi_j^a \gamma_{sk} + 2\phi_{kj} \gamma_s^a) \nabla_a R.$$

By interchanging the indices s and j at (4.2) and adding each other, we have, from (2.22)

$$(4.3) \quad \nabla_s H_{jk} + \nabla_j H_{sk} = -\frac{1}{4(n+1)} (2\rho_k \gamma_{sj} - \rho_s \gamma_{jk} - \rho_j \gamma_{sk} + 3(\phi_{kj} \nabla_s R + \phi_{ks} \nabla_j R)),$$

where we put $\phi_k^a \nabla_a R = \rho_k$.

Since $\tilde{\rho}_s = \phi_s^r \rho_r = -\nabla_s R$ by virtue of (2.22), the equation (4.3) is rewritten as

$$(4.4) \quad \nabla_s H_{jk} + \nabla_j H_{sk} = -\frac{1}{4(n+1)} (2\rho_k \gamma_{sj} - \rho_s \gamma_{jk} - \rho_j \gamma_{sk} + 3(\tilde{\rho}_s \phi_{jk} + \tilde{\rho}_j \phi_{sk})).$$

If we put $u_{jk} = -4(n+1)H_{jk} = -4(n+1)\phi_j^a R_{ak}$, we have

$$(4.5) \quad \nabla_s u_{jk} + \nabla_j u_{sk} = 2\rho_k \gamma_{sj} - \rho_s \gamma_{jk} - \rho_j \gamma_{sk} + 3(\tilde{\rho}_s \phi_{jk} + \tilde{\rho}_j \phi_{sk}).$$

Therefore we can find that $-4(n+1)\phi_j^i R_{ti}$ is a ηK -conformal Killing tensor field with its associated vector $\phi_k^a \nabla_a R$. Thus we obtain the following.

THEOREM 4.1. *In a $(2n + 1)$ -dimensional cosymplectic manifold with vanishing cosymplectic Bochner curvature tensor and non-constant scalar curvature, $-4(n + 1)\phi_j^a R_{ai}$ is a ηK -conformal Killing tensor field.*

COROLLARY 4.2. *In a 3-dimensional cosymplectic manifold with non-constant scalar curvature, $-8\phi_j^a R_{ai}$ is a ηK -conformal Killing tensor field.*

REMARK. Eum[3] has found an example of η -conformal Killing tensor field in a cosymplectic manifold of constant curvature with respect to γ_{ji} .

5. The converse case

In the last section, we consider the converse case of the prior section. That is, we will investigate how the cosymplectic Bochner curvature tensor roles when the cosymplectic manifold admits a ηK -conformal Killing tensor field.

At first, we suppose that $-4(n + 1)\phi_j^a R_{ai}$ is a ηK -conformal Killing tensor field in a $(2n + 1)$ -dimensional cosymplectic manifold.

Hereafter, we put $\phi_j^a R_{ai} = H_{ji}$, then from the definition, we have

$$(5.1) \quad \begin{aligned} & -4(n + 1)(\nabla_k H_{ji} + \nabla_j H_{ki}) \\ & = 2\rho_i \gamma_{kj} - \rho_k \gamma_{ji} - \rho_j \gamma_{ki} + 3(\tilde{\rho}_k \phi_{ji} + \tilde{\rho}_j \phi_{ki}), \end{aligned}$$

where $\rho_k = \frac{1}{(n+1)}(i(\eta)\rho)\eta_k - 2\nabla^a H_{ak}$ and $\tilde{\rho}_k = \phi_k^a \rho_a$ and $i(\eta)\rho = \eta^a \rho_a$.

If the scalar field $i(\eta)\rho$ vanishes identically, then we call ρ_a the horizontal vector field.

In the first place, contracting (5.1) with η^k and using (2.1) and (2.23), we have $(i(\eta)\rho)\gamma_{ji} = 0$. By the way, in [7], we have proved that γ_{ji} is positive definite for any vector $X^i \neq (\eta_t X^t)\eta^i$. Thus we have

LEMMA 5.1. *The associated vector field of a ηK -conformal Killing tensor field $-4(n + 1)\phi_j^a R_{ai}$ in a $(2n + 1)$ -dimensional cosymplectic manifold is horizontal.*

Transvecting (5.1) with g^{jk} and taking account of Lemma 5.1, (2.5) and (2.9), we obtain

$$(5.2) \quad \rho_k = \phi_k^a \nabla_a R.$$

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Hence we have from (2.22)

$$(5.3) \quad \tilde{\rho}_k = -\nabla_k R.$$

If we substitute (5.2) and (5.3) into (5.1), then it follows that

$$(5.4) \quad \begin{aligned} & -4(n+1)(\nabla_k H_{ji} + \nabla_j H_{ki}) \\ & = 2\phi_i^a \gamma_{kj} \nabla_a R - \phi_k^a \gamma_{ji} \nabla_a R - \phi_j^a \gamma_{ki} \nabla_a R - 3(\phi_{ji} \nabla_k R + \phi_{ki} \nabla_j R). \end{aligned}$$

Furthermore, making use of (2.14), (2.19), (2.22) and (5.4), the following equation holds good:

$$(5.5) \quad \begin{aligned} & -4(n+1)(\nabla_i R_{jk} - 2\nabla_j R_{ki}) \\ & = (2\phi_{kj} \phi_i^a - \phi_{ki} \phi_j^a + \gamma_{ji} \delta_k^a - 3\phi_k^a \phi_{ji} + 3\gamma_{ki} \delta_j^a) \nabla_a R. \end{aligned}$$

Interchanging the indices k and i at (5.5) and subtracting the resulting equation from (5.5), we find

$$(5.6) \quad \begin{aligned} \nabla_k R_{ij} - \nabla_i R_{kj} - \frac{1}{4(n+1)} (\gamma_k^a \gamma_{ij} - \gamma_i^a \gamma_{kj} + \phi_k^a \phi_{ij} \\ - \phi_i^a \phi_{kj} - 2\phi_j^a \phi_{ki}) \nabla_a R = 0, \end{aligned}$$

from which, we obtain $B_{kji} = 0$, where we have used (3.15).

Thus we have from (3.14) the following.

THEOREM 5.2. *Let M be a $(2n+1)$ -dimensional cosymplectic manifold admitting a ηK -conformal Killing tensor field $-4(n+1)\phi_j^a R_{ai}$. Then the cosymplectic Bochner curvature tensor satisfies the harmonic condition, that is*

$$\nabla_a B_{kji}^a = 0.$$

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