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EVALUATION SUBGROUPS AND CELLULAR EXTENSIONS OF CW–COMPLEXES

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1. Introduction

D.H.Gottlieb [1, 2] studied the subgroups $G_n(X)$ of homotopy groups $\pi_n(X)$. In [5, 7, 10], the authors introduced subgroups $G_n(X,A)$ and $G_n^{Rel}(X,A)$ of $\pi_n(X)$ and $\pi_n(X,A)$ respectively and showed that they fit together into a sequence

$$\begin{aligned} \cdots \to G_n(A) \xrightarrow{i_*} G_n(X,A) \xrightarrow{j_*} G_n^{Rel}(X,A) \xrightarrow{\partial} \\ \cdots \to G_1^{Rel}(X,A) \to G_0(A) \to G_0(X,A) \end{aligned}$$

where i_*, j_* and ∂ are restrictions of the usual homomorphisms of the homotopy sequence

$$\cdots \xrightarrow{\partial} \pi_n(A) \xrightarrow{i_*} \pi_n(X) \xrightarrow{j_*} \pi_n(X,A) \to \cdots \to \pi_0(A) \to \pi_0(X).$$

This sequence are called the G-sequence of (X,A) in [6,10]. We also showed it is exact when the inclusion $i : A \to X$ has a left homotopy inverse or is homotopic to a constant. From the G-sequence of a CW-pair, we defined the ω-homology of a CW-pair and identified the relation between the ω-homology and the exactness of G-sequence of a CW-pair.

In this paper, we introduce the concept of a wedge compressible pair and study its properties, especially the relation between the wedge

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compressibility and ω-homology or finite exactness of G-sequences. For any space and any number k, we construct a k-exact G-sequence. As its application, we calculate the evaluation subgroups of cellular extensions of some CW-complexes and the generalized evaluation subgroup of a pair which consists of a CW-complex and its cellular extension. Finally we show that if the ω-homology of a pair is not trivial then the homotopy group of a space is not trivial in some dimension.

2. Definitions and notations

In this paper, all spaces are connected CW-complexes, all topological pairs are CW-pairs and all subspaces mentioned contain the same base point as their total spaces. Let $[A^A]$ be the subspace of the function space X^A which consists of $f \in X^A$ such that $f(A) \subset A$. Since $[A^A]$ is homeomorphic to A^A, we use A^A instead of $[A^A]$. Let us take s_0 as the base point of S^n and x_0 as the base point of X and its subspaces. We use the same notation ω for the evaluation maps of X^X and X^A into X at the base point x_0 and use i as the inclusion map. And we use X^X and X^A as the path-components of the function spaces X^X and X^A containing 1_X and i respectively. We leave the base points out of the notation for the homotopy groups when the simplication will not lead to confusion. Here we recall the definitions of the evaluation subgroups $G_n(X), G_n(X,A)$ of the homotopy group $\pi_n(X)$ and the relative evaluation subgroups $G_n^{Rel}(X,A)$ of the relative homotopy groups $\pi_n(X,A)$ for a CW-pair (X,A) in [2] , [9] and [6,10]. There are two equivalent definitions for each of these subgroups.

DEFINITION 1. $G_n(X) = \omega_*(\pi_n(X^X, 1_X)) = \{[f] \in \pi_n(X) | \exists$ map $H : X \times I^n \longrightarrow X$ such that $[H|_{x_0 \times I^n}] = [f]$ and $H|_{X \times u} = 1_X$ for $u \in \partial I^n\}$[2].

DEFINITION 2. $G_n(X,A) = \omega_*(\pi_n(X^A, i)) = \{[f] \in \pi_n(X) | \exists$ map $H : A \times I^n \longrightarrow X$ such that $[H|_{x_0 \times I^n}] = [f]$ and $H|_{A \times u} = i$ for $u \in \partial I^n\}$[9].

DEFINITION 3. $G_n^{Rel}(X,A) = \omega_* \pi_n(X^A, A^A, i) = \{[f] \in \pi_n(X,A) | \exists$ map $H : (X \times I^n, A \times \partial I^n) \longrightarrow (X,A)$ such that $[H|_{x_0 \times I^n}] = [f]$ and $H|_{X \times u} = 1_X$ for $u \in J^{n-1}\}$[6, 10].

These groups fit together into a sequence

$$\xrightarrow{\partial^{n+1}} G_n(A) \xrightarrow{i_*^n} G_n(X,A) \xrightarrow{j_*^n} G_n^{Rel}(X,A) \xrightarrow{\partial^n}$$
$$\cdots \to G_1^{Rel}(X,A) \xrightarrow{\partial^1} G_0(A) \to G_0(X,A)$$

which are called the *G-sequence* of (X,A) . We can think of the G-sequence as a chain complex.

DEFINITION 4. If (X,A) is a CW-pair, the *ω-homology*

$$H_*^\omega(X,A) = \{H_{n+1}^{g\omega}(X,A), H_{n+1}^{r\omega}(X,A), H_n^{a\omega}(X,A)\}_{n\geq 0}$$

of (X,A) is defined to be

$$H_{n+1}^{g\omega}(X,A) = Ker\ j_*^{n+1}/Im\ i_*^{n+1},$$
$$H_{n+1}^{r\omega}(X,A) = Ker\ \partial^{n+1}/Im\ j_*^{n+1},$$
$$H_n^{a\omega}(X,A) = Ker\ i_*^n/Im\ \partial^{n+1}$$

for $n \geq 0$. [6]

3. Finite G-sequences and ω-homology

We first define the wedge compressibility and find some equivalent conditions to wedge compressility.

DEFINITION 5. A CW-pair (X,A) is said to be *k-wedge compressible* if for every positive integer $n \leq k$ and for every map $H : A \times S^n \to X$ such that $H|_{A\times s_0} = i$, there exists a map $H' : A \times S^n \to A$ such that $H'|_{A\times s_0} = 1_A$ and $i \circ H'|_{x_0\times S^n}$ is homotopic to $H|_{x_0\times S^n}$ relative to s_0.

We say that a CW-pair (X,A) is *wedge compressible* if (X,A) is k-wedge compressible for every positive integer k. By definition, if (X,A) is k-wedge compressible then (X,A) is r-wedge compressible for every positive integer $r \leq k$.

EXAMPLE 1. If X is contractible, then (X, A) is wedge compressible for any subspace A. Let $H : A\times S^n \to X$ is a map such that $H|_{A\times s_0} = i$. Since X is contractible, H is homotopic to c_{x_0}(the constant map at x_0). Define $H' : A\times S^n \to A$ by $H'(a,s) = a$, then $i\circ H'|_{x_0\times S^n}$ is homotopic to c_{x_0} and hence is homotopic to $H|_{x_0\times S^n}$.

If we consider the commutative diagram

$$\begin{array}{ccc} \pi_n(A^A, 1_A) & \xrightarrow{\bar{i}_*} & \pi_n(X^A, i) \\ \downarrow \omega_* & & \downarrow \omega_* \\ \pi_n(A, x_0) & \xrightarrow{i_*} & \pi_n(X, x_0) \end{array}$$

where $\bar{i}(g) = ig$ for $g \in A^A$,then the k-wedge compressibility is explained by the above diagram as follows:

THEOREM 1. *A CW-pair (X, A) is k-wedge compressible if and only if $\pi_n(X^A, i) = Im\bar{i}_* + Ker\ \omega_*$ for every positive integer $n \le k$.*

Proof. Suppose that (X, A) is k-wedge compressible. Let $n \le k$ and $[h]$ be any element of $\pi_n(X^A, i)$. Then there is a map $H : A \times S^n \to X$ given by $H(a, s) = h(s)(a)$. Since $H|_{A\times s_0} = i$, there exists a map $H' : A\times S^n \to A$ such that $H'|_{A\times s_0} = 1_A$ and $i\circ H'|_{x_0\times S^n}$ is homotopic to $H|_{x_0\times S^n}$ relative to s_0. Define $h' : (S^n, s_0) \to (A^A, 1_A)$ be a map given by $h'(s)(a) = H'(a, s)$. Then we have

$$\omega_*(\bar{i}_*[h']) = i_*\omega_*[h'] = [i\circ H'|_{x_0\times S^n}] = [H|_{x_0\times S^n}] = \omega_*[h]$$

This implies that $[h]$ - $\bar{i}_*[h']$ belongs to $Ker\ \omega_*$. Therefore $[h] = \bar{i}_*[h'] + [g]$ for some $[g]$ in $Ker\ \omega_*$.

Conversely, let $H : A \times S^n \to X$ be a map such that $H|_{A\times s_0} = i$. Then the adjoint map $h : (S^n, s_0) \to (X^A, i)$ given by $h(s)(a) = H(a, s)$ is represented by $[h] = \bar{i}_*[h'] + [g]$ for $[h'] \in \pi_n(A^A, 1_A)$ and $[g] \in Ker\ \omega_*$. If we define a map $H' : A \times S^n \to A$ given by $H'(a, s) = h'(s)(a)$, then we have $H'|_{A\times s_0} = 1_A$ and

$$[i\circ H'|_{x_0\times s^n}] = i_*\omega_*[h'] = \omega_*\bar{i}_*[h'] = \omega_*[h] = [H|_{x_0\times S^n}].$$

Thus $i \circ H'|_{x_0 \times S^n}$ is homotopic to $H|_{x_0 \times S^n}$ relative to s_0. Hence (X, A) is k-wedge compressible.

In Lemma 2[3], it is shown that if X is aspherical, then X^A is also aspherical and $\omega_* : \pi_1(X^A, i) \to \pi_1(X, x_0)$ is a monomorphism. Therefore $\omega_* : \pi_n(X^A, i) \to \pi_n(X, x_0)$ is a monomorphism for any positive integer n.

COROLLARY 2. *Let X be aspherical. A CW-pair (X, A) is k-wedge compressible if and only if $i_* : \pi_1(A^A, i_A) \to \pi_n(X^A, i)$ is surjective for every positive integer $n \leq k$.*

COROLLARY 3. *Let X be aspherical. If a CW-pair (X, A) is 1-wedge compressible,then (X, A) is wedge compressible.*

The following theorem shows a relation between the wedge compressibility and the evaluation subgroups.

THEOREM 4. *A CW-pair (X, A) is k-wedge compressible if and only if $i_*(G_n(A)) = G_n(X, A)$ for every positive integer $n \leq k$.*

Proof. Suppose that (X, A) is k-wedge compressible.
Since $i_*(G_n(A)) \subset G_n(X, A)$, it is sufficient to show that $G_n(X, A) \subset i_*(G_n(A))$. Let $[f] \in G_n(X, A)$. Then there exists a map $H : A \times S^n \to X$ such that $H|_{A \times s_0} = i$ and $[H|_{x_0 \times S^n}] = [f]$. Since (X, A) is k-wedge compressible, there is a map $H' : A \times S^n \to A$ such that $H'|_{A \times s_0} = 1_A$ and $i \circ H'|_{x_0 \times S^n}$ is homotopic to $H|_{x_0 \times S^n}$ relative to s_0. Thus we have $i_*[H'|_{x_0 \times S^n}] = [i \circ H'|_{x_0 \times S^n}] = [H|_{x_0 \times S^n}] = [f]$. Therefore we obtain $[f] = i_*[H'|_{x_0 \times S^n}] \in i_*(G_n(A))$.

Conversely, let $n \leq k$ and $H : A \times S^n \to X$ be a map such that $H|_{A \times s_0} = i$. Then $[H|_{x_0 \times S^n}] \in G_n(X, A) = i_*(G_n(A))$. Therefore, there is an element $[g] \in G_n(A)$ such that $i_*[g] = [H|_{x_0 \times S^n}]$. Since $[g] \in G_n(A)$, there is a map $H' : A \times S^n \to A$ such that $H'|_{A \times s_0} = 1_A$ and $H'|_{x_0 \times S^n}$ is homotopic to g relative to s_0. Therefore $i \circ H'|_{x_0 \times S^n}$ is homotopic to $H|_{x_0 \times S^n}$ relative to s_0 and hence (X, A) is k-wedge compressible.

By Theorem 4 and the definition of ω-homology, we have the following corollary.

COROLLARY 5. *If a CW-pair (X,A) is k-wedge compressible, then the ω-homology $H_n^{g\omega}(X,A)$ is trivial for every positive integer $n \leq k$.*

Proof. Let (X,A) be k-wedge compressible. Then by Theorem 4, $i_*(G_n(A)) = G_n(X,A)$. Since $i_*(G_n(A)) \subset Ker\ j_* \subset G_n(X,A)$, we have $i_*(G_n(A)) = Ker\ j_*$, where i_* and $j_*$are homomorphisms in the G-sequence. Thus $H_n^{g\omega}(X,A)$ is trivial for every positive integer $n \leq k$.

COROLLARY 6. *If X is a k-connected space, then (X,A) is k-wedge compressible for any subspace A of X.*

Proof. Since X is k-connected, we have $\pi_n(X) = 0$ for $n \leq k$. This implies $i_*(G_n(A)) = G_n(X,A) = 0$ for each $n \leq k$. By Theorem 4, (X,A) is k-wedge compressible.

The following theorem shows that for a given space A and a given positive integer k, we can construct a proper super space X such that (X,A) is k-wedge compressible.

THEOREM 7. *Let A be a CW-complex and X be an m-cellular extension of A for $m > 1+dimA$. Then the pair (X,A) is $(m-dimA-1)$-wedge compressible.*

Proof. Let $n \leq m - \dim A - 1$ and $H : A \times S^n \to X$ be the map such that $H|_{A\times s_0} = i$. Then H may be considered as a pair map $H : (A\times S^n, A\times s_0) \to (X,A)$. Since $\pi_k(X,A) = 0$ for $k < m$ (Proposition 16.4, in [4]), $A\times S^n - A\times s_0$ has cells in dimension $< m$ and $n < m$, there is a map $H' : A\times S^n \to A$ such that $i\circ H'$ is homotopic to H relative to $A\times s_0$ (Theorem 16.6, in [4]). Let the homotopy $F : (A\times S^n\times I, A\times s_0\times I) \to (X,A)$ be the map such that $F|_{A\times S^n\times 0} = H$, $F|_{A\times S^n\times 1} = H'$ and $F|_{A\times s_0\times I} = H|_{A\times s_0} = H'|_{A\times s_0}$. Thus $H'(a,s_0) = H(a,s_0) = a$ and $i\circ H'|_{x_0\times S^n}$ is homotopic to $H|_{x_0\times S^n}$ relative to s_0 by the homotopy $F' = F|_{x_0\times S^n\times I} : S^n\times I \to X$. Thus (X,A) is $(m - \dim A - 1)$-wedge compressible.

By Theorem 7 and Corollary 5, we have the following corollary

COROLLARY 8. *Let A be a CW-complex and X be an m-cellular extension of A for $m > 1 + dimA$. Then $H_n^{g\omega}(X,A) = 0$ for every $n \leq m - dimA - 1$.*

DEFINITION 6. A CW-pair (X, A) is said to have *k-exact G-sequence* if the following finite G-sequence is exact;

$$G_k(A) \xrightarrow{i_*} G_k(X, A) \xrightarrow{j_*} G_k^{Rel}(X, A) \xrightarrow{\partial} \cdots \to G_1^{Rel}(X, A) \to G_0(A) \to G_0(X, A).$$

A CW-pair (X, A) has the exact G-sequence if and only if it has the k-exact G-sequence for every positive integer k. If (X, A) has the k-exact G-sequence, then it has the n-exact G-sequence for $n \leq k$.

THEOREM 9. *Suppose (X, A) is a CW-pair with cells in $X - A$ only in dimension $> k$. Then (X, A) is k-wedge compressible if and only if (X, A) has the k-exact G-sequence.*

Proof. Consider the following finite G-sequence of (X, A)

$$G_k(A) \xrightarrow{i_*} G_k(X, A) \xrightarrow{j_*} G_k^{Rel}(X, A) \to \cdots \to G_1^{Rel}(X, A) \to G_0(A) \to G_0(X, A).$$

If (X, A) is k-wedge compressible, then $i_*(G_n(A)) = G_n(X, A)$ for $n \leq k$. Since (X, A) is a CW-pair with cells in $X - A$ only in dimension $> k$, we have $\pi_n(X, A) = 0$ for $n \leq k$ and hence $G_n^{Rel}(X, A) = 0$. Thus $Ker\ j_* = G_n(X, A) = i_*(G_n(A)) = Im\ i_*$. Moreover, $\partial = 0$ and i_* is monomorphism in the finite G-sequence, so the finite G-sequence is exact until $G_1(X, A)$. Finally, we will show the finite G-sequence is exact at $G_1^{Rel}(X, A)$. Consider the following ladder

$$\begin{array}{ccccccc} \to \pi_1(X^A) & \xrightarrow{j_*} & \pi_1(X^A, A^A) & \xrightarrow{\partial} & * & \longrightarrow & * \\ \downarrow \omega_* & & \downarrow \omega_* & & \downarrow \omega_* & & \downarrow \omega_* \\ \to G_1(X, A) & \xrightarrow{j_*} & G_1^{Rel}(X, A) & \xrightarrow{\partial} & * & \longrightarrow & *, \end{array}$$

then j_* and ω_* are surjective. Therefore $j_* : G_1(X, A) \xrightarrow{j_*} G_1^{Rel}(X, A)$ is surjective and hence the proof is complete.

Conversely, if (X, A) has the k-exact G-sequence, then $i_*(G_n(A)) = G_n(X, A)$ by exactness and $G_n^{Rel}(X, A) = 0$ for $1 \leq n \leq k$. Thus by Theorem 4, (X, A) is k- wedge compressible.

COROLLARY 10. *If (X, A) is a CW-pair with cells (in $(X - A)$) only in dimension $> k$, then ω-homology*

$$H_n^{\omega}(X, A) = \{H_{n+1}^{g\omega}(X, A), H_{n+1}^{r\omega}(X, A) \text{ and } H_n^{a\omega}(X, A)\}$$

is trivial for $n < k$ if and only if (X, A) is k-wedge compressible.

The following two equivalent corollaries follow from Theorem 9.

COROLLARY 11. *Let A be a CW-complex and X be an m-cellular extension of A where $m > 1 + dimA$. Then (X, A) has $(m - dimA - 1)$-exact G-sequence.*

Proof. Since $X - A$ has cells in dimension $\geq m$ and (X, A) is $(m - dimA - 1)$-wedge compressible, (X, A) has the $(m - dimA - 1)$-exact G-sequence by Theorem 9.

COROLLARY 12. *Let (X, A) be the pair in Corollary 11. Then the ω-homology*

$$H_n^{\omega}(X, A) = \{H_{n+1}^{g\omega}(X, A), H_{n+1}^{r\omega}(X, A) \text{ and } H_n^{a\omega}(X, A)\}$$

is trivial for $n < m - dimA - 1$.

DEFINITION 7. A CW-pair (X, A) is ***k*-absolute wedge compressible** if for every subcomplex B of A with the same base point x_0, every $n \leq k$, and every map $H : B \times S^n \to X$ such that $H|_{B \times s_0} = i$, there exists a map $H' : B \times S^n \to A$ such that $H'|_{B \times s_0} = i$ and $i \circ H'|_{x_0 \times S^n}$ is homotopic to $H|_{x_0 \times S^n}$relative tos_0.

By definition, if (X, A) is k-absolute wedge compressible, then (X, A) is k-wedge compressible. A triple (X, A, B) is called a CW-triple if (X, A), (X, B) and (A, B) are CW-pairs[6].

LEMMA 13. *Let (X, A, B) be a CW-triple. If (X, A) and (A, B) are k-absolute wedge compressible, then (X, B) is k-absolute wedge compressible.*

Proof. Let C be a subcomplex of B and $H : C \times S^n \to X$ be a map such that $H|_{C \times s_0} = i$. Since (X, A) is a k-absolute wedge compressible, there is a map $H_1 : C \times S^n \to A$ such that $H_1|_{C \times s_0} = i_{CA} : C \to A$ and

$i_{AX} \circ H_1|_{x_0 \times S^n}$ is homotopic to $H|_{x_0 \times S^n}$ relative to s_0. Moreover, since (A, B) is k-absolute wedge compressible, there is a map $H_2 : C \times S^n \to B$ such that $H_2|_{C \times s_0} = i_{CB} : C \to B$ and $i_{BA} \circ H_2|_{x_0 \times S^n}$ is homotopic to $H_1|_{x_0 \times S^n}$ relative to s_0. Now $i_{BX} \circ H_2|_{x_0 \times S^n} = i_{AX} \circ i_{BA} \circ H_2|_{x_0 \times S^n}$ and hence is homotopic to $i_{AX} \circ H_1|_{x_0 \times S^n}$. Therefore $i_{BX} \circ H_2|_{x_0 \times S^n}$ is homotopic to $H|_{x_0 \times S^n}$ relative to s_0.

THEOREM 14. *Let* $X = A \cup e^m \cup e^{2m-dim A} \cup \cdots \cup e^{km-(k-1)dim A}$ *for* $m > 1 + dimA$. *Then* (X, A) *is* $(m - dimA - 1)$*-absolute wedge compressible.*

Proof. We can prove that $(A \cup e^m, A)$ is $(m - dimA - 1)$-absolute wedge compressible. Moreover, $(A \cup e^m \cup e^{2m-dimA}, A \cup e^m)$ is $(2m - dimA) - dim(A \cup e^m) - 1 (= m - dimA - 1)$-absolute wedge compressible. Thus $(A \cup e^m \cup e^{2m-dimA}, A)$ is $(m - dimA - 1)$- absolute wedge compressible. By induction on k, (X, A) is $(m - dimA - 1)$ absolute wedge compressible.

The following corollary follows from Theorem 4, Lemma 13 and Theorem 14.

COROLLARY 15. *Let* (X, A) *be the pair in Theorem 14. Then the following equivalent statements hold:*

(1) $i_*(G_n(A)) = G_n(X, A)$ *for* $n \leq m - dimA - 1$,

(2) (X, A) *has the* $(m - dimA - 1)$*-exact* G*-sequence,*

(3) *the* ω*-homology* $H_n^{\omega}(X, A) = \{H_{n+1}^{g\omega}(X, A), H_{n+1}^{r\omega}(X, A)$ *and* $H_n^{a\omega}(X, A)\}$ *is trivial for* $n < m - dimA - 1$.

4. Applications

As an application of the results of the preceding section, we calculate the evaluation subgroups and the generalized evaluation subgroups (or ω-homology) of some pairs which consist of CW-complexs and their cellular extensions.

Let A be a CW-complex and X be an m-cellular extension of A for $m > 1 + dimA$.

THEOREM 16. *Let X be an m-cellular extension of A and $n < m-$ dim A. If $G_n(A) = 0$, then $G_n(X, A) = 0$ and hence $G_n(X) = 0$.*

Proof. Since $i_* : G_n(A) \to G_n(X, A)$ is an epimorphism by Theorem 4 and 7. Since $G_n(X) \subset G_n(X, A)$, $G_n(A) = 0$ implies $G_n(X, A) = 0 = G_n(X)$.

COROLLARY 17. *$G_n(S^n \cup_\alpha e^m) = 0$ if n is even and $m > 2n$.*

Proof. By Theorem 5.4 in [2], $G_n(S^n) = 0$ if n is even. Thus $G_n(S^n \cup_\alpha e^m) = 0$ by Theorem 16.

COROLLARY 18. *If CP^n is the n-dimensional complex projective plane and $m > 2n + 3$, then $G_2(CP^n \cup_\alpha e^m, CP^n) = 0$ and hence $G_2(CP^n \cup_\alpha e^m) = 0$*

Proof. By Theorem 3.6 in [5], $G_2(CP^n) = 0$ for all n. Thus $G_2(CP^n \cup_\alpha e^m, CP^n) = 0$ by Theorem 16.

THEOREM 19. *If X is an m-cellular extension of a G-space A, then $G_n(X, A) = \pi_n(X)$ for $n < m$.*

Proof. Since $X - A$ has the only one cell e^m, we obtain $\pi_k(X, A) = 0$ for $k < m$. Thus $i_* : \pi_n(A) \to \pi_n(X)$ is an epimorphism in the homotopy sequence of (X, A). Since A is a G-space, we have $G_n(A) = \pi_n(X)$ and

$$\pi_n(X) = i_*(\pi_n(A)) = i_*(G_n(A)) \subset G_n(X, A) \subset \pi_n(X).$$

Therefore we have $G_n(X, A) = \pi_n(X)$.

Since every H-space is a G-space, S^k is a G-space if $k = 1, 3, 7$. Thus for $m > k$ we have

COROLLARY 20. *Let $k = 1, 3, 7$ and $m > k$. Then $G_n(S^k \cup_\alpha e^m, S^k) = \pi_n(S^k \cup_\alpha e^m)$ for $n < m$.*

THEOREM 21. *Let $m >$ dim$A + 1$ and $f : S^{m-1} \to A$ be a map. Then $G_n(A \cup_f e^m, A)$ is isomorphic to $G_n(A)$ for $n < m -$ dim$A - 1$.*

Proof. The pair $(A \cup_f e^m, A)$ has the $(m-$dim$A-1)$ exact G-sequence by Corollary 11. Moreover, $G_n^{Rel}(A \cup_f e^m, A) = 0$ for $n < m$ because this is a subgroup of $\pi_n(A\cup_f e^m, A)$. Thus $i_* : G_n(A) \to G_n(A\cup_f e^m, A)$ is an isomorphism in the G-sequence for $n < m - \dim A - 1$. Therefore $G_n(A \cup_f e^m, A)$ is isomorphic to $G_n(A)$ for $n < m - \dim A - 1$.

COROLLARY 22.

$$G_n(S^n \cup_f e^m, S^n) \cong \begin{cases} 0 & \text{if } n \text{ is even} \\ 2\mathbf{Z} & \text{if } n \text{ is odd, } n \neq 1,3 \text{ or } 7 \\ \mathbf{Z} & \text{if } n = 1,3 \text{ or } 7. \end{cases}$$

if $m > 2n + 1$.

As an application of ω-homology, we investigate the relation between ω-homology and a homotopy group in cellular extensions of CW-complexes.

THEOREM 23. *If* $f : S^{m-1} \to A$ *is null-homotopic, then* $(A\cup_f e^m, A)$ *has the exact* G*-sequence and hence the* ω*-homology* $H_*^\omega(A \cup_f e^m, A)$ *is trivial.*

Proof. In [7,10], we have proved that if the inclusion $i : A \to X$ has a left homotopy inverse, then the pair (X,A) has the exact G- sequence. Thus it is sufficient to show that the inclusion $i : A \to A \cup_f e^m$ has a left homotopy inverse. Since f is null-homotopic, there is an extension $F : CS^{m-1} \to A$ of f and $1_A \amalg F : A \amalg CS^{m-1} \to A$ induces a map $r : A \cup_f e^m \to A$ such that $r \circ i = 1_A$. Therefore r is a left homotopy inverse of i and hence the pair $(A \cup_f e^m, A)$ has the exact G-sequence.

In [6], we have showed that if the inclusion $i : A \to A \cup_f e^m$ is null-homotopic, then the pair $(A \cup_f e^m, A)$ has the exact G-sequence. So we can conclude that if i is null-homotopic or f is null-homotopic, then the ω-homology $H_*^\omega(A \cup_f e^m, A)$ is trivial.

COROLLARY 25. *If* $H_*^\omega(A \cup_f e^m, A) \neq 0$*, then* $\pi_{m-1}(A) \neq 0$.

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