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ISOMETRIES WITH SMALL BOUND ON $C^1(X)$ SPACES

KIL-WOUNG JUN AND YANG-HI LEE

1. Introduction

For a locally compact Hausdorff space X, we denote by $C_0(X)$ the Banach space of all continuous complex valued functions defined on X which vanish at infinity, equipped with the usual sup norm. In case X is compact, we write $C(X)$ instead of $C_0(X)$. A well-known Banach-Stone theorem states that the existence of an isometry between the function spaces $C_0(X)$ and $C_0(Y)$ implies X and Y are homeomorphic. D. Amir [1] and M. Cambern [2] independently generalized this theorem by proving that if $C_0(X)$ and $C_0(Y)$are isomorphic under an isomorphism T satisfying $\|T\|\|T^{-1}\| < 2$, then X and Y must also be homeomorphic.

We denote by $C^1(X)$ the space of continuously differentiable functions on X with the Σ-norm given by $\|f\| = \sup_{t\in X} |f(t)|+\sup_{t\in X} |f'(t)|$. And we denote by $C^1(X)_p$ the space of continuously differentiable functions on X with the norm given by $\|f\|_p = \sup_{t\in X} |f(t)| + p\sup_{t\in X} |f'(t)|, p > 0$.

K. Jarosz [4] conjectured that; Is there a positive ϵ such that for any compact subsets X, Y of the real line R and $\|T\|\|T^{-1}\| < 1+\epsilon$ implies that X and Y are homeomorphic? When the norms of $C^1(X)$ and $C^1(Y)$ are given by the C-norms, Cambern and Pathak[3] proved the existence of ϵ in the additional assumption $\|T\|_\infty\|T^{-1}\|_\infty < \infty$. When the norms of $C^1(X)$ and $C^1(Y)$ are given by the M-norms, Pathak and Vasavada[6] proved the existence of ϵ in the additional assumption $\|T\|_\infty\|T^{-1}\|_\infty < \infty$. In this note we investigate the Jarosz conjecture when the norms of $C^1(X)$ and $C^1(Y)$ are given by the Σ-norms.

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2. The results

THEOREM 1. *Let X and Y be compact subsets of R and $X \subset [a,b]$ and $Y \subset [c,d]$. If T is a linear map from $C^1(X)$ onto $C^1(Y)$ which satisfies*

(i) *if $f'(t) \equiv 0$ then $(Tf)'(t) \equiv 0$,*
(ii) $\|fg\| \leq \|TfTg\| \leq (1+\epsilon)^2\|fg\|$,
(iii) $\|f\| \leq \|Tf\| \leq (1+\epsilon)\|f\|$, *and*
(iv) $\epsilon < \min(\frac{1}{49}, \frac{1}{2(|b-a|+1)}, \frac{1}{2(|d-c|+1)})$,
then X and Y are homeomorphic.

Before proving the theorem let us prove three lemmas.

LEMMA 1. *Let X and Y be as in Theorem 1. Let T be a map from $C^1(X)$ onto $C^1(Y)$ which satisfies the condition* (ii) *in Theorem 1. If $\sup_{t\in X}|f(t)| < k$ and $\|f\| = 1$ then $\sup_{s\in Y}|Tf(s)| \leq (1+\epsilon)\sqrt{-k^2+2k}$.*

Proof.

$$\begin{aligned}(\sup_{s\in Y}|Tf(s)|)^2 &\leq \|TfTf\| \leq (1+\epsilon)^2\|f^2\| \\ &= (1+\epsilon)^2(\sup_{t\in X}|f^2(t)| + \sup_{t\in X}|(f^2)'(t)|) \\ &= (1+\epsilon)^2(\sup_{t\in X}|f^2(t)| + 2\sup_{t\in X}|f(t)f'(t)|) \\ &\leq (1+\epsilon)^2(-k^2+2k),\end{aligned}$$

and this completes the proof.

LEMMA 2. *Let X, Y, T be as in Theorem 1. If $f \in C^1(X)$ then $\frac{5}{7}\sup_{t\in X}|f'(t)| \leq \sup_{s\in Y}|(Tf)'(s)|$.*

Proof. Let $f \in C^1(X)$ and $\sup_{t\in X}|f'(t)| \neq 0$. Then $f' \in C(X)$. We can extend f' to g' on $C([a,b])$ such that $g'|_X = f'$ and $\sup_{t'\in[a,b]}|g'(t')| = \sup_{t\in X}|f'(t)|$. Let $g(t') = \int_a^{t'} g'(x)dx$. Then $\sup_{t\in X}|g|_X(t)| \leq |b-a|\sup_{t\in X}|(g|_X)'(t)|$ and $\|g|_X\| \leq (|b-a|+1)\sup_{t\in X}|(g|_X)'(t)|$. Take $m \in X$ such that $|f'(m)| = \sup_{t\in X}|f'(t)|$. Fix $k < \frac{1}{4}$ and choose $h' \in C([a,b])$ such that $h'(m) = 2$, $h'(t') = 0$, for $t' \notin [m - \frac{k}{2}\sup_{t\in X}|f'(t)|, m + \frac{k}{2}\sup_{t\in X}|f'(t)|]$ and $0 \leq h'(t') \leq 2$. Let $i'(t') = g'(t')h'(t')$

and $i(t') = \int_a^{t'} i'(x)dx$. Then $i|_X \in C^1(X)$ and $\sup_{t\in X} |i|_X(t)| \leq k \sup_{t\in X} |(i|_X)'(t)|$. Hence

$$\begin{aligned} 2\sup_{t\in X} |(g|_X)'(t)| = & \sup_{t\in X} |(i|_X)'(t)| \leq \|i|_X\| \\ &= \sup_{t\in X} |i|_X(t)| + \sup_{t\in X} |(i|_X)'(t)| \\ &\leq (1+k) \sup_{t\in X} |(i|_X)'(t)| \\ &= 2(1+k) \sup_{t\in X} |(g|_X)'(t)|. \end{aligned}$$

By Lemma 1,
$2\sup_{t\in X} |(g|_X)'(t)|(1-(1+\epsilon)\sqrt{-k^2+2k}) \leq \sup_{s\in Y} |(T(i|_X))'(s)|$.
(1)

$$\begin{aligned} &\|T((g-i)|_X)\| \\ \geq\ & \sup_{s\in Y} |T(g|_X)(s)| - \sup_{s\in Y} |T(i|_X)(s)| - \sup_{s\in Y} |(T(g|_X))'(s)| \\ &+ \sup_{s\in Y} |(T(i|_X))'(s)| \\ \geq\ & \sup_{s\in Y} |T(g|_X)(s)| - 2(1+k)(1+\epsilon)\sqrt{-k^2+2k} \sup_{t\in X} |(g|_X)'(t)| \\ &+ 2(1-(1+\epsilon)\sqrt{-k^2+2k}) \sup_{t\in X} |(g|_X)'(t)| - \sup_{s\in Y} |(T(g|_X))'(s)|. \end{aligned}$$

And,
(2)

$$\begin{aligned} \|T((g-i)|_X)\| &\leq (1+\epsilon)\|g|_X - i|_X\| \\ &\leq (1+\epsilon)(\sup_{t\in X} |g|_X(t)| + \sup_{t\in X} |i|_X(t)| + \sup_{t\in X} |(g|_X)'(t)|) \\ &\leq \|g|_X\| + \frac{1}{2} \sup_{t\in X} |(g|_X)'(t)| + (1+\epsilon)2k \sup_{t\in X} |(g|_X)'(t)|. \end{aligned}$$

If $\sup_{s\in Y} |(T(g|_X))'(s)| < \frac{5}{7}(1 - \frac{28}{5}(1+\epsilon)\sqrt{-k^2+2k})$ $\sup_{t\in X} |(g|_X)'(t)|$, then by (1) $\|T(g|_X - i|_X)\| \geq \|T(g|_X)\| +$ $\frac{4}{7}\sup_{t\in X} |(g|_X)'(t)|$. From (2) $\|g|_X\| + \frac{1}{2}\sup_{t\in X} |(g|_X)'(t)| + (1+\epsilon)2k$ $\sup_{t\in X} |(g|_X)'(t)| \geq \|T(g|_X)\| + \frac{4}{7}\sup_{t\in X} |(g|_X)'(t)|$. Since k is arbitrary, this contradicts the fact $\|Tg\| \geq \|g\|$. From $f'(t) = (g|_X)'(t)$

we know that $(Tf)'(s) = (T(g|_X))'(s)$. Hence $\sup_{s\in Y}|(Tf)'(s)| \geq \frac{5}{7}\sup_{t\in X}|f'(t)|$.

LEMMA 3. *Let X, Y, T be as in Theorem 1. If $f \in C^1(X)$ then* $\frac{5}{7}\sup_{t\in X}|f'(t)| \leq \sup_{s\in Y}|(Tf)'(s)| \leq \frac{7}{5}(1+\epsilon)\sup_{t\in X}|f'(t)|$.

Proof. By Lemma 2, $\frac{5}{7}\sup_{t\in X}|f'(t)| \leq \sup_{s\in Y}|(Tf)'(s)|$. Hence if $(Tf)'(s) \equiv 0$ then $f'(t) \equiv 0$. Replacing T by $(1+\epsilon)T^{-1}$ and applying Lemma 1 and 2, we know that $\frac{5}{7}\sup_{s\in Y}|f'(s)| \leq (1+\epsilon)\sup_{t\in X}|(T^{-1}f)'(t)|$. From this we have $\frac{5}{7}\sup_{s\in Y}|(Tf)'(s)| \leq (1+\epsilon)\sup_{t\in X}|f'(t)|$.

Proof of Theorem 1. Let $g(t') = \int_{a_i}^{t'} g'(x)dx$ for $g' \in C([a,b])$. Let S be a map from $C([a,b])$ into $C^1(X)$ defined by $Sg' = g|_X$. Choose a map U from $C(X)$ into $C([a,b])$ such that $(Uf)|_X = f$ for all $f \in C(X)$. If $f \in C^1(X)$ then $f' \in C(X)$ and $(SUf')' = f'$. Let S' be a map from $C^1(Y)$ onto $C(Y)$ defined by $S'g(s) = g'(s)$. Since $(SU(\alpha f + \beta g))' - \alpha(SUf)' - \beta(SUg)' \equiv 0$, $(T(SU(\alpha f + \beta g) - \alpha(SUf) - \beta(SUg)))' \equiv 0$ i.e., $S'TSU(\alpha f + \beta g)) - \alpha(S'TSUf) - \beta(S'TSUg))' \equiv 0$. Hence $S'TSU$ is a linear map from $C(X)$ into $C(Y)$.

Let $T' : C(X) \to C(Y)$ be defined by $T'f'(s) = (Tf)'(s)$ for all $f \in C^1(X)$. Since $f' = (SUf')'$, $T'f' = (Tf)' = (TSUf')' = S'TSUf'$. Therefore by Lemma 3, T' is a onto linear map such that $\|T'\| \leq \frac{7}{5}(1+\epsilon)$ and $\|T'^{-1}\| \leq \frac{7}{5}$. Therefore $\|T'\|\|T'^{-1}\| \leq \frac{49}{25}(1+\epsilon)$. By Amir theorem [1] X and Y are homeomorphic.

COROLLARY 1. *Let X and Y be compact subsets of R. Let T be a linear map from $C^1(X)$ onto $C^1(Y)$ which satisfies*

(i) *if $f'(t) \equiv 0$ then $(Tf)'(t) \equiv 0$*

(ii) $\|fg\| \leq \|TfTg\| \leq (1+\epsilon)^2\|fg\|$.

If ϵ is sufficiently small, then X and Y are homeomorphic.

Proof. From (i) and (ii) we have $1 \leq \|T1T1\| = (\sup_{s\in Y}|T1(s)|)^2 = \|T1\|^2 \leq (1+\epsilon)^2$, and so $1 \leq \|T1\| \leq (1+\epsilon)$. By (ii) $\|T^{-1}1\cdot 1\| \leq \|T1\cdot 1\| \leq (1+\epsilon)^2\|T^{-1}1\cdot 1\|$. Hence $\|T^{-1}1\| \leq \|T1\| \leq (1+\epsilon)$ and $\frac{1}{(1+\epsilon)^2} \leq \frac{1}{(1+\epsilon)^2}\|T1\| \leq \|T^{-1}1\|$. For any $g \in C^1(X)$, $\|g\| \leq \|T1Tg\| \leq \|T1\|\|Tg\| \leq (1+\epsilon)\|Tg\|$ and $\|Tg\| \leq (1+\epsilon)^2\|T^{-1}1\cdot g\| \leq$

$(1+\epsilon)^2\|T^{-1}1\|\|g\| \le (1+\epsilon)^3\|g\|$. Hence $\frac{1}{(1+\epsilon)}\|g\| \le \|Tg\| \le (1+\epsilon)^3\|g\|$.

If ϵ is sufficiently small, then $(1+\epsilon)T$ satisfies the conditions of Theorem 1.

THEOREM 2. *Let X and Y be compact subsets of R and $X \subset \cup_{i=1}^n[a_i, b_i]$ $(a_i < b_i < a_{i+1})$ and $\max_i\{|b_i - a_i|\} < k$. $Y \subset \cup_{j=1}^m[c_j, d_j]$ $(c_j < d_j < c_{j+1})$ and $\max_j\{|d_j - c_j|\} < k$. If T is a linear map from $C^1(X)$ onto $C^1(Y)$ which satisfies*

(i) $(Tf)'(t) \equiv 0$ *iff* $f'(t) \equiv 0$,

(ii) $\|f\| \le \|Tf\| \le (1+\epsilon)\|f\|$, *and*

(iii) $k < \frac{(4-\sqrt{10})}{6}$ *and* $\epsilon < 6k^2 - 8k + 1$,

then X and Y are homeomorphic.

Proof. Let $\sup_{t\in X}|f(t)| < k$ and $\|f\| = 1$. If $\sup_{s\in Y}|Tf(s)| > 3k$, choose $G'(s') \in C(\cup_{j=1}^m[c_j, d_j])$ such that $\sup_{s'\in\cup_{j=1}^m[c_j,d_j]}|G'(s')| = \sup_{s\in Y}|(Tf)'(s)|$ and $G'(s) = (Tf)'(s)$ for all $s \in Y$. Let $G(s') = \int_{c_j}^{s'} G'(y)dy$ for all $s' \in [c_j, d_j]$. Then we have

$$\begin{aligned}
\sup_{s'\in\cup_{j=1}^m[c_j,d_j]}|G(s')| &\le \max_j \sup_{s'\in[c_j,d_j]}\int_{c_j}^{s'}|G'(y)|dy \\
&\le \max_j \sup_{s'\in[c_j,d_j]}\{\sup_{t\in[c_j,d_j]}|G'(t)||c_j - d_j|\} \\
&\le k\max_j \sup_{s'\in[c_j,d_j]}|G'(s')| \\
&= k\sup_{s\in Y}|G'(s)|.
\end{aligned} \tag{3}$$

Hence we have $\sup_{s\in Y}|Tf(s)| - \sup_{s\in Y}|G(s)| > 2k - k\epsilon$ and $\sup_{s\in Y}|Tf(s)| > \sup_{s\in Y}|G(s)|$. Therefore $\|G|_Y\| < (1+\epsilon) - 2k + k\epsilon$, and hence,

$$\|T^{-1}(G|_Y)\| < (1+\epsilon) - 2k + k\epsilon. \tag{4}$$

By assumption $\sup_{t\in X}|f'(t) - (T^{-1}(G|_Y))'(t)| = 0$.
Therefore $\|T^{-1}(G|_Y)\| \ge \sup_{t\in X}|(T^{-1}(G|_Y))'(t)| = \sup_{t\in X}|f'(t)| \ge 1 - k$. This contradicts (4). Hence $(1-3k)\sup_{t\in X}|f'(t)| \le \sup_{s\in Y}|(Tf)'(s)| \le \frac{1+\epsilon}{1-k}\sup_{t\in X}|f'(t)|$.

Let $f \in C^1(X)$ and $\sup_{t\in X} |f'(t)| \neq 0$.
Choose $g'(t') \in C(\cup_{i=1}^n [a_i, b_i])$ such that $\sup_{t'\in \cup_{i=1}^n [a_i,b_i]} |g'(t')|$ $= \sup_{t\in X} |f'(t)|$ and $g'(t) = f'(t)$ for all $t \in X$. Let $g(t') = \int_{a_i}^{t'} g'(x)dx$ for all $t' \in [a_i, b_i]$. By the similar method in (3), $\sup_{t\in X} |g|_X(t)| \leq \sup_{t'\in \cup_{i=1}^n [a_i,b_i]} |g(t')| \leq k \sup_{t\in X} |(g|_X)'(t)|$. Hence $(1-3k)\sup_{t\in X} |(g|_X)'(t)| \leq \sup_{s\in Y} |(T(g|_X))'(s)| < \frac{1+\epsilon}{1-k} \sup_{t\in X} |(g|_X)'(t)|$, and so $(1-3k)\sup_{t\in X} |f'(t)| \leq \sup_{s\in Y} |(Tf)'(s)| \leq \frac{1+\epsilon}{1-k} \sup_{t\in X} |f'(t)|$.

As in the proof of Theorem 1, if $T' : C(X) \to C(Y)$ is defined by $T'f'(t) = (Tf)'(t)$ for all $f \in C^1(X)$, then $\|T'\|\|T'^{-1}\| \leq \frac{1+\epsilon}{(1-3k)(1-k)}$. By Amir theorem [1] X and Y are homeomorphic.

COROLLARY 2. *Let X be a Cantor set and k, ϵ, Y, T be as in Theorem 2. Then X and Y are homeomorphic.*

Proof. For any $k > 0$ there exsits $\cup_{i=1}^n [a_i, b_i]$ such that $X \subset \cup_{i=1}^n [a_i, b_i]$ $(a_i < b_i < a_{i+1})$ and $\max_i\{|b_i - a_i|\} < k$. By Theorem 2, X and Y are homeomorphic.

THEOREM 3. *Let X and Y be compact subsets of R and $X \subset \cup_{i=1}^n [a_i, b_i]$ $(a_i < b_i < a_{i+1})$ and $\max_i\{|b_i - a_i|\} < k$. $Y \subset \cup_{j=1}^m [c_j, d_j]$ $(c_j < d_j < c_{j+1})$ and $\max_j\{|d_j - c_j|\} < k$. If T is a linear map from $C^1(X)_p$ onto $C^1(Y)_p$ which satisfies*

(i) $f'(t) \equiv 0$ iff $(Tf)'(t) \equiv 0$,

(ii) $\|f\|_p \leq \|Tf\|_p < (1+\epsilon)\|f\|_p$, and

(iii) $pk < (4-\sqrt{10})/6$ and $\epsilon < 6(pk)^2 - 8pk + 1$,

then X and Y are homeomorphic.

Proof. Let S be a map from $C^1(X)_p$ onto $C^1(pX)$ defined by $Sf(px) = f(x)$ and S' be a map from $C^1(Y)_p$ onto $C^1(pY)$ defined by $S'g(py) = g(y)$. Since $\|Sf\| = \sup_{px\in pX} |Sf(px)| + \sup_{px\in pX} |Sf'(px)| = \sup_{x\in X} |f(x)| + p\sup_{x\in X} |f'(x)| = \|f\|_p$, S is a linear isometric map. Simiraly so is S'. Let T_1 be a map from $C^1(pX)$ onto $C^1(pY)$ defined by $T_1 f(py) = S'TS^{-1}f(py)$. Then T_1 is a linear map and $\|f\| \leq \|T_1 f\| < (1+\epsilon)\|f\|$. Note that $pX \subset \cup_{i=1}^n [pa_i, pb_i]$, $\max_i\{|pb_i - pa_i|\} < pk$, $pY \subset \cup_{j=1}^m [pc_j, pd_j]$ and $\max_i\{|pc_j - pd_j|\} < pk$ and apply Theorem 2. Hence pX and pY are homeomorphic. This completes the proof.

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KIL-WOUNG JUN
DEPARTMENT OF MATHEMATICS, CHUNGNAM NATIONAL UNIVERSITY, TAEJON 305-764, KOREA

YANG-HI LEE
DEPARTMENT OF MATHEMATICS EDUCATION, KONGJU NATIONAL TEACHERS COLLEGE, KONGJU 314-060, KOREA