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ELLIPTIC SYSTEMS INVOLVING COMPETING INTERACTIONS WITH NONLINEAR DIFFUSIONS

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1. Introduction

Our interest is to study the existence of positive solutions to the following elliptic system involving competing interaction

$$(1)\qquad \begin{cases} -\varphi(x,u,v)\Delta u = uf(x,u,v) \\ -\psi(x,u,v)\Delta v = vg(x,u,v) \\ \dfrac{\partial u}{\partial n} + \kappa u = 0 \quad \text{on } \partial\Omega \\ \dfrac{\partial v}{\partial n} + \sigma v = 0 \end{cases}$$

in a bounded region Ω in $\mathbf{R}^n$ with a smooth boundary, where the diffusion terms φ, ψ are strictly positive nondecreasing function, and κ, σ are positive constants. Also we assume that the growth rates f, g are C^1 monotone functions. The variables u, v may represent the population densities of the interacting species in problems from ecology, microbiology, immunology, etc.

In this paper, we give the existence theorem of positive solutions to the above elliptic systems for competing interactions arising on the biological models. This model is characterized by the monotonicity of f, g, i.e., two species are in competing interaction if each of their relative growth functions is decreasing in the other. Refer to [9] for details.

In [1], we provided sufficient and necessary conditions of existence conditions for the predator-prey interactions. In that paper, we proved that the existence of positive solutions can be characterized by the spectral property of a certain operator of Schrödinger type.

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There has been several results for system (1) under Dirichlet or Neumann boundary conditions where the diffusion terms $\varphi \equiv \psi \equiv$ constants. See [6], [7], [8], [5], [10] etc. For those work, topological index theory and decomposed operator method have been successful in proving the existence theorems. In this paper, we employ the decomposed operator motivated by [3] to prove the existence theorem of positive solutions of system (1).

In section 2, we collect some known lemmas from [1]. In section 3, we give sufficient conditions of the existence of positive solutions for competing interactions.

From this paper and [1], we conclude that if the system involves predator-prey or competing interactions, then the existence of positive solutions can be characterized by the sign of the principal eigenvalues of certain differential operators which are determined by given systems. Therefore the positive coexistence for a 2×2 system involving the above two interactions is determined by the spectral properties of this system, i.e., the existence of positive solutions is affected by the shape and size of the domain.

2. Preparations

In this section, we state some known lemmas which will serve as the basic tools for the arguments used to prove our results. Throughout this paper we will consider problems in the space $\mathbf{X} = C(\bar{\Omega})$, where Ω is a bounded region in $\mathbf{R}^n$. Let $r(T)$ denote the spectral radius of a linear operator T and $\lambda_1(A)$ the first eigenvalue of a suitable operator A.

Let $\mathbf{X}$ be a Banach space and let F be a strongly positive nonlinear compact operator on $\mathbf{X}$ such that $F(0) = 0$.

LEMMA 1. *Assume $F'(0)$ exists with $r(F'(0)) > 1$. If there is no $\mu \in (0,1]$ in any neighborhood of which the equation $u = \mu F u$ has a solution u as $\|u\| \to \infty$, then F has a positive fixed point u such that $Fu = u$ in the positive cone $\mathbf{K}$ of $\mathbf{X}$.*

Proof. See Theorem 13.2 [2].

OBSERVATION 1. *Let $a(x) \geq \delta_0 > 0$ and $b(x) \in L^\infty(\Omega)$. Also let P be positive constant such that $P + b(x) > 0$ for $x \in \Omega$. Then*
(i) $\lambda_1(a(x)\Delta + b(x)) > 0 \Leftrightarrow r[(-a(x)\Delta + P)^{-1}(P + b(x))] > 1$
(ii) $\lambda_1(a(x)\Delta + b(x)) < 0 \Leftrightarrow r[(-a(x)\Delta + P)^{-1}(P + b(x))] < 1$
(iii) $\lambda_1(a(x)\Delta + b(x)) = 0 \Leftrightarrow r[(-a(x)\Delta + P)^{-1}(P + b(x))] = 1$
where λ_1 is the first eigenvalue under Robin boundary condition.

OBSERVATION 2. *Suppose that $a \in C^1(\bar{\Omega})$ and $a(x) \geq \delta_0 > 0$ and $b(x) \in L^\infty(\Omega)$. Let $\tau > 0$ be a constant. Then there exists $u > 0$, $u \in C^1(\bar{\Omega})$, such that for a unique $\lambda_1 > 0$,*

$$\begin{cases} -a(x)\Delta u + b(x)u = \lambda_1 u \\ \dfrac{\partial u}{\partial n} + \tau u = 0 \quad on\ \partial\Omega. \end{cases}$$

Moreover, λ_1 is increasing in $\frac{b(x)}{a(x)}$.

We define the classes F and $G \subset C(\Omega \times \mathbf{R}^+)$ as follows:
Let $L \in \mathbf{R}^+$. $f \in F$ if and only if $f \in C(\Omega \times \mathbf{R}^+)$ and
(F1) $f \in C^1$ in ξ, $f_\xi(x,\xi) < 0$ in $\Omega \times \mathbf{R}^+$ and $|f_\xi(x,\xi)| \leq L$ for $(x,\xi) \in \Omega \times [0, c_0]$.
(F2) $f(x,0) > 0$ and $f(x,\xi) < 0$, where $(x,\xi) \in \Omega \times (c_0, \infty)$ for some constant $c_0 > 0$.
(F3) $f(x,\xi)$ is concave down on the set of (x,ξ) where $f(x,\xi) < 0$.

Let $\varphi = \varphi(x,\xi) \in C(\Omega \times \mathbf{R})$ and be C^1-function in ξ. Then $\varphi \in G$ if and only if φ satisfies
(G1) $\varphi(x,\xi) \geq \delta > 0$ for some constant δ and $\xi \in \mathbf{R}^+$, $x \in \Omega$.
(G2) $\varphi(x,\xi)$ is nondecreasing and concave down in $\xi \in \mathbf{R}^+$.

For the proofs of the next three lemmas can be found in [1].

LEMMA 2. *Let $f \in F$ and $\varphi \in G$. Then $\frac{f(x,u)}{\varphi(x,u)}$ is decreasing in u for $x \in \Omega$.*

Consider the following equation,

$$(2) \qquad \begin{cases} -\varphi(x,u)\Delta u = uf(x,u) \quad \text{in } \Omega \\ \dfrac{\partial u}{\partial n} + \kappa u = 0 \quad \text{on } \partial\Omega \end{cases}$$

where $\kappa > 0$ is a constant.

In [1], we proved that the linearization of the equation (2) at $u = 0$ is

$$(3) \qquad \begin{cases} -\varphi(x,0)\Delta\cdot = \cdot f(x,0) \\ \dfrac{\partial\cdot}{\partial n} + \kappa\cdot = 0 \quad \text{on } \partial\Omega \end{cases}$$

Let $\mathbf{K} = C(\bar{\Omega})^+$ be the positive cone of the ordered Banach space $C(\bar{\Omega})$. We define the ordered interval $[[u_1, u_2]] := \{u \in C(\bar{\Omega}) : u_1 \leq u \leq u_2 \quad \text{for } u_1,\, u_2 \in C(\bar{\Omega})\}$.

In the next lemma, $\lambda_1(A)$ denotes the first eigenvalue of an operator A under the boundary condition $\frac{\partial u}{\partial n} + \kappa u = 0$.

LEMMA 3. *Let $f \in F$ and $\varphi \in G$.*

(i) *If $\lambda_1(\varphi(x,0)\Delta + f(x,0)) > 0$, then the equations (2) have a unique positive solution in $C^2(\bar{\Omega})$.*

(ii) *If $\lambda_1(\varphi(x,0)\Delta + f(x,0)) \leq 0$, then $u \equiv 0$ is the only nonnegative solution of (2).*

According to Lemma 3, the equation (2) has a unique positive solution. We denote it by $u_{\varphi,f}$. Let u_{φ_n,f_n} be the unique positive solution of

$$(4) \qquad \begin{cases} -\varphi_n(x,u)\Delta u = u f_n(x,u) \\ \dfrac{\partial u}{\partial n} + \kappa u = 0 \quad \text{on } \partial\Omega. \end{cases}$$

LEMMA 4. *If $f \in F$ and $\varphi \in G$, then*

(i) *$(\varphi, f) \mapsto u_{\varphi,f}$ is a continuous mapping of $G \times F \to C^{1,\alpha}(\Omega \times \mathbf{R}^+)$ for some $\alpha \in (0,1)$.*

(ii) *If $\frac{f_1}{\varphi_1} \geq \frac{f_2}{\varphi_2} \not\equiv \frac{f_1}{\varphi_1}$, for $x \in \Omega$, then either $u_{\varphi_1,f_1} > u_{\varphi_2,f_2}$ or $u_{\varphi_1,f_1} \equiv u_{\varphi_2,f_2} \equiv 0$.*

3. Existence theorem

In this section, we investigate the existence of positive solutions of the system (1):

$$\begin{cases} -\varphi(x,u,v)\Delta u = uf(x,u,v) \\ -\psi(x,u,v)\Delta v = vg(x,u,v) \\ \dfrac{\partial u}{\partial n} + \kappa u = 0 \quad \text{on } \partial\Omega \\ \dfrac{\partial v}{\partial n} + \sigma v = 0 \end{cases}$$

For the competing interaction, we make the following assumptions on the system (1).

(C1) $f, g \in C^1(\bar{\Omega} \times \mathbf{R}^+ \times \mathbf{R}^+)$ satisfy

$$\begin{cases} f_u(x,u,v) < 0 & f_v(x,u,v) < 0 \quad \text{for } u, v > 0 \\ g_u(x,u,v) < 0 & g_v(x,u,v) < 0 \quad \text{for } u, v > 0 \end{cases}$$

Furthermore, $f_v, g_u \neq 0$ and all partial derivatives are uniformly bounded on $\bar{\Omega} \times \mathbf{R}^+ \times \mathbf{R}^+$.

(C2) There exist positive constants C_1, C_2 such that

$$\begin{cases} f(x,u,0) < 0 \quad \text{for } u > C_1 \\ g(x,0,v) < 0 \quad \text{for } v > C_2 \end{cases}$$

(C3) Let $\varphi(x,u,v)$, $\psi(x,u,v) \in C(\bar{\Omega} \times \mathbf{R}^+ \times \mathbf{R}^+)$ and $\varphi(x,\cdot,v)$, $\psi(x,u,\cdot) \in G$ for fixed u, $v \in \mathbf{R}^+$.

The assumption (C1) describes how these species u, v interact with each other, while the assumption (C2) indicates that the model under consideration is logistic.

In this section, let $\lambda_{1,\kappa}(A)$, $\lambda_{1,\sigma}(A)$ denote the first eigenvalue of an operator A under the boundary conditions $\frac{\partial\cdot}{\partial n} + \kappa\cdot = 0$ and $\frac{\partial\cdot}{\partial n} + \sigma\cdot = 0$ respectively.

REMARK. By Lemma 3, if $\lambda_{1,\kappa}(\varphi(x,0,0)\Delta + f(x,0,0)) > 0$ when $\varphi \in G$, then the equation

$$\begin{cases} -\varphi(x,u,0)\Delta u = uf(x,u,0) \\ \dfrac{\partial u}{\partial n} + \kappa u = 0 \quad \text{on } \partial\Omega \end{cases}$$

has a unique positive solution u_0.

Similarly, if $\lambda_{1,\sigma}(\psi(x,0,0)\Delta + g(x,0,0)) > 0$ when $\psi \in G$, then

$$\begin{cases} -\psi(x,0,v)\Delta v = vg(x,0,v) \\ \dfrac{\partial v}{\partial n} + \sigma v = 0 \quad \text{on } \partial\Omega \end{cases}$$

has a unique positive solution v_0.

Before we give the existence theorem, we show *a priori* bounds of a positive solution (u,v).

LEMMA 5. *Assume $\lambda_{1,\kappa}(\varphi(x,0,0)\Delta + f(x,0,0)) > 0$ and $\lambda_{1,\sigma}(\psi(x,0,0) + g(x,0,0)) > 0$.*

If (u,v) is a strictly positive solution to (1), then we have the following inequalities:

$$0 < u(x) < u_0(x) < C_1, \qquad 0 < v(x) < v_0(x) < C_2$$

for $x \in \Omega$.

Proof. Let u be a positive solution of

$$(5) \qquad \begin{cases} -\varphi(x,u,v)\Delta u = uf(x,u,v) \\ \dfrac{\partial u}{\partial n} + \kappa u = 0 \quad \text{on } \partial\Omega. \end{cases}$$

Also let u_0 be a positive solution of

$$(6) \qquad \begin{cases} -\varphi(x,u,0)\Delta u = uf(x,u,0) \\ \dfrac{\partial u}{\partial n} + \kappa u = 0 \quad \text{on } \partial\Omega. \end{cases}$$

Since $\frac{f(x,u,0)}{\varphi(x,u,0)} \geq \frac{f(x,u,v)}{\varphi(x,u,v)}$ for $v \geq 0$ by Lemma 2, u is a lower solution of (6). Next one can show $u_0 \leq C_1$. In fact, if $\max_{x\in\Omega} u(x) = u(x_0) > C_1$ for some $x_0 \in \Omega$, then $f(x_0, u(x_0), 0) < 0$. Thus $0 \leq -\varphi(x_0, u(x_0), 0)\Delta u(x_0) = u(x_0) f(x_0, u(x_0), 0) < 0$. This contradiction shows $u_0 \leq C_1$. Let $\bar{u} = \max(C_1 + 1, \max_{x\in\Omega} u)$. Then since $\bar{u}$ is an upper solution to (6), there exists u_1 such that $u \leq u_1 \leq \bar{u}$. By the uniqueness of solution to (6), we have $u_1 = u_0 \leq C_1$. Then the strong maximal principle implies $0 < u < u_0 < C_1$. Similarly, one can show $0 < v < v_0 < C_2$.

Now we give sufficient conditions for the existence of positive solutions of the system (1).

THEOREM. *Assume (C1)-(C3).*

(a) *If* $\lambda_{1,\kappa}(\varphi(x,0,0)\Delta + f(x,0,0)) \leq 0$ *or* $\lambda_{1,\sigma}(\psi(x,0,0)\Delta + g(x,0,0)) \leq 0$, *then the system (1) has no positive solution.*

(b) *Assume* $\lambda_{1,\kappa}(\varphi(x,0,0)\Delta + f(x,0,0)) > 0$ *and* $\lambda_{1,\sigma}(\psi(x,0,0)\Delta + g(x,0,0)) > 0$.

If $\lambda_{1,\kappa}(\varphi(x,0,v_0)\Delta + f(x,0,v_0)) > 0$ *and* $\lambda_{1,\sigma}(\psi(x,u_0,0)\Delta + g(x,u_0,0)) > 0$, *then (1) has a positive solution.*

Proof. (a) Suppose $(\bar{u},\bar{v})$ is a positive solution of system (1). Then $\bar{u} \not\equiv 0$ and $(\varphi(x,\bar{u},\bar{v})\Delta + f(x,\bar{u},\bar{v}))\bar{u} = 0$. Thus according to the last part of Observation 2, we have $\lambda_{1,\kappa}(\varphi(x,0,0)\Delta + f(x,0,0)) > \lambda_{1,\kappa}(\varphi(x,\bar{u},0)\Delta + f(x,\bar{u},0)) > \lambda_{1,\kappa}(\varphi(x,\bar{u},\bar{v})\Delta + f(x,\bar{u},\bar{v}))$, a contradiction. So the system has no positive solution. The other case is similar.

(b) Since $\lambda_{1,\kappa}(\varphi(x,0,0)\Delta + f(x,0,0)) > 0$ and $\varphi \in G$, by Lemma 3, the equation

$$(7)\qquad \begin{cases} -\varphi(x,u,v(x))\Delta u = uf(x,u,v(x)) \\ \dfrac{\partial u}{\partial n} + \kappa u = 0 \quad \text{on } \partial\Omega \end{cases}$$

has a unique positive solution u. Then, according to the Lemma 4, we may define the operator $T : C(\bar{\Omega}) \to C(\bar{\Omega})$ such that $u = Tv$ is a unique positive solution to the equation (7). Note that by Lemma 4, it is easy to see that T is a continuous operator and T is strictly monotone. In fact, in case that u and v are in competition, T is decreasing in v. Substituting u by Tv in the other equation, we have

$$(8) \qquad \begin{cases} -\psi(x,Tv,v)\Delta v = vg(x,Tv,v) \\ \dfrac{\partial v}{\partial n} + \sigma v = 0 \ \text{ on } \partial\Omega \end{cases}$$

We define the operator $A : C(\bar{\Omega}) \to C(\bar{\Omega})$ by considering the equation

$$(9) \qquad \begin{cases} -\psi(x,Tv,w)\Delta w + Pw = vg(x,Tv,v) + Pv \\ \dfrac{\partial w}{\partial n} + \sigma w = 0 \ \text{ on } \partial\Omega. \end{cases}$$

Denote this by $w = Av = (-\psi(x,Tv,\cdot)\Delta \cdot + P\cdot)^{-1}[vg(x,Tv,v) + Pv]$ where P is a positive constant. Now we are looking for a fixed point v of the operator A in the positive cone $\mathbf{K}$. Then (Tv, v) is a positive solution of (1). To this end, we apply nonlinear fixed point theorem (see Lemma 1). Write $A_\theta = \theta A$. Suppose v_θ is a fixed point of A_θ where $v_\theta \in \mathbf{K}$. Then

$$(10) \qquad -\psi\Big(x, Tv, \frac{v_\theta}{\theta}\Big)\Delta v_\theta = \theta v_\theta g(x, Tv_\theta, v_\theta) + (\theta - 1)Pv_\theta.$$

Suppose $v_\theta \not\equiv 0$. Let $v_\theta(x_0) = \max_{x\in\bar{\Omega}} v_\theta(x) > 0$ for some $x_0 \in \bar{\Omega}$. Then $x_0 \in \Omega$ and at the point x_0 the left side of (10) is nonnegative and $(\theta - 1)Pv_\theta(x_0) \leq 0$. Thus we have $g(x_0, Tv_\theta(x_0), v_\theta(x_0)) \geq 0$. Since the system involves competing interaction, $g(x, Tv, v) \leq g(x, 0, v)$ implies that $0 \leq g(x_0, Tv_\theta(x_0), v_\theta(x_0)) \leq g(x, 0, v_\theta(x_0))$. So by the assumption (C2), we have $v_\theta(x_0) \leq C_2$. Therefore there is an *a priori* bound for the positive fixed point of A_θ. Also observe that $A'(0) = (-\psi(x, u_0, 0)\Delta + P)^{-1}(g(x, u_0, 0) + P)$ and $r(A'(0)) > 1$ by Observation 1. Since A is a strongly positive compact operator from $C(\bar{\Omega})$ to $C(\bar{\Omega})$, A has a fixed point v in $\mathbf{K}$ by Lemma 1. Note that $u = Tv > 0$. In fact, suppose $Tv \equiv 0$. Then $v = v_0$, so $0 = \lambda_{1,\sigma}(\psi(x,0,v)\Delta + g(x,0,v)) \leq \lambda_{1,\sigma}(\psi(x,0,0)\Delta + g(x,0,0)) \leq 0$, a contradiction. Thus $Tv > 0$. Therefore (Tv, v) is a positive solution of the system (1).

Next, we formulate the corresponding results for particular cases of the system (1). Consider the following competing system

$$(11) \qquad \begin{cases} -(c_1+u)\Delta u = u(e_1 - a_1 v - b_1 u) \\ -(c_2+v)\Delta v = v(e_2 - a_2 u - b_2 v) \\ \dfrac{\partial u}{\partial n} + \kappa u = 0 \quad \text{on } \partial\Omega \\ \dfrac{\partial v}{\partial n} + \sigma v = 0 \end{cases}$$

where all constants a_i, b_i, c_i are positive for $i = 1, 2$.

Then the following corollary is immediate. Let λ_1 denote the first eigenvalue of $-\Delta$.

COROLLARY. *Assume $\frac{e_1}{c_1} > \lambda_1$ and $\frac{e_2}{c_2} > \lambda_1$. Then the system (11) has a positive solution if $\frac{b_2 e_1 - a_1 e_2}{b_2 c_1} > \lambda_1$ and $\frac{b_1 e_2 - a_2 e_1}{b_1 c_2} > \lambda_1$.*

Proof. As we mentioned in Remark, by the assumptions $\frac{e_1}{c_1} > \lambda_1$ and $\frac{e_2}{c_2} > \lambda_1$, there exist positive solutions u_0 to the equation

$$\begin{cases} -(c_1+u)\Delta u = u(e_1 - b_1 u) \\ \dfrac{\partial u}{\partial n} + \kappa u = 0 \quad \text{on } \partial\Omega \end{cases}$$

and v_0 to the equation

$$\begin{cases} -(c_2+v)\Delta = v(e_2 - b_2 v) \\ \dfrac{\partial v}{\partial n} + \sigma v = 0 \quad \text{on } \partial\Omega. \end{cases}$$

If $u_0 > \frac{e_1}{b_1}$, then we have a contradiction by using the general maximum principle. Thus $0 < u_0 \leq \frac{e_1}{b_1}$. Also $\|v_0\|_\infty \leq \frac{e_2}{b_2}$. Therefore $\lambda_{1,\kappa}(c_1\Delta + e_1 - a_1 v_0) \geq \lambda_{1,\kappa}(c_1\Delta + e_1 - a_1\frac{e_2}{b_2}) > 0$. Similarly, $\lambda_{1,\sigma}(c_2\Delta + e_2 - a_2 u_0) > 0$. Now apply Theorem to get the result.

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