NON-OVERLAPPING CONTROL SYSTEMS ON $Aff(\mathbb{R})$

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1. Introduction

Let $G$ be a Lie group with Lie algebra $L(G)$ and let $\Omega$ be a non-empty subset of $L(G)$. If $\Omega$ is interpreted as the set of controls, then the set of elements attainable from the identity for the system $\Omega$ is a subsemigroup of $G$. A system $\Omega$ is called a non-overlapping control system if any element attainable for $\Omega$ is only attainable at one time. In [1], we showed that a compact, convex generating non-overlapping control systems on a connected solvable Lie group must be contained in $X + E$, where $E$ is a subspace of codimension one containing the commutator and the homomorphism from the attainable semigroup into $\mathbb{R}^+$ extends continuously to the whole group. In this paper, we show that in $aff(\mathbb{R})$, the unique two dimensional non-abelian Lie algebra, a non-overlapping control system must be contained in $X + [aff(\mathbb{R}), aff(\mathbb{R})]$ and the homomorphism from the attainable semigroup into $\mathbb{R}^+$ extends continuously to $Aff(\mathbb{R})$.

Let $G$ be a Lie group with Lie algebra $L(G)$. Normally we identify $L(G)$ with the set of right invariant vector fields on $G$. For a non-empty subset $\Omega$ of $L(G)$, we consider the control system on $G$ given by the differential equation

$$x'(t) = U(t)(x(t)),$$

where $U$ belongs to the class $\mathcal{U}(\Omega)$ of measurable functions from $\mathbb{R}^+ = [0, \infty)$ into $\Omega$ which are locally bounded. A solution of (*) is an absolutely continuous function $x(\cdot)$ defined on $\mathbb{R}^+$ such that the equation

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(*) holds almost everywhere. In [8], for $U \in \mathcal{U}(\Omega)$, it is known that there exists a unique solution $x(\cdot)$ on $G$ of the initial value problem

$$x'(t) = U(t)(x(t)), \quad x(0) = g.$$  

We denote this solution by $\pi(g, \cdot, U)$, i.e., $\pi(g, 0, U) = g$ and $\pi(g, t, U) = x(t)$ for all $t \in \mathbb{R}^+$. If there exists $U \in \mathcal{U}(\Omega)$ such that $h = \pi(g, t, U)$, then we say that $h$ is attainable from $g$ at time $t$ for the system $\Omega$. The set of such elements attainable from $g$ at time $t$ for the system $\Omega$ (resp. using only piecewise constant controls into $\Omega$) is denoted $A(g, t, \Omega)$ (resp. $A_{pc}(g, t, \Omega)$). We also employ the notation

$$A(g, T, \Omega) = \bigcup_{0 \leq t \leq T} A(g, t, \Omega)$$

$$A(g, \Omega) = \bigcup_{0 \leq t < \infty} A(g, t, \Omega)$$

The set $A(g, \Omega)$ is called the attainability set from $g$. From the right invariance of the control system, $A(g, T, \Omega) = A(e, T, \Omega)g$, and $A(g, \Omega) = A(e, \Omega)g$. Thus we restrict our attention to the attainability set at the identity.

For a non-empty subset $\Omega$ of $L(G)$, we have the one-parameter semigroup of sets $t \rightarrow A(e, t, \Omega)$. That is, $A(e, s, \Omega)A(e, t, \Omega) = A(e, s + t, \Omega)$ for all $s, t \in \mathbb{R}^+$ ([5],[8]). This implies that the attainability set at the identity is a subsemigroup of $G$ and $S(\Omega) := A(e, \Omega)$ is called the attainable semigroup for $\Omega$. In this paper, we say that $\Omega$ generates $L(G)$ if $L(G)$ is the smallest subalgebra containing $\Omega$.

2. Non-overlapping control systems on $\text{Aff}(\mathbb{R})$

We start by summarizing the useful properties concerning control systems on Lie groups. The following results could be found in [5].

**Proposition 1.** Let $\Omega$ be a non-empty subset of $L(G)$ and $\hat{\Omega}$ be its closed convex hull. Then

1. $A_{pc}(e, t, \Omega) = \{\exp(t_1X_1) \cdots \exp(t_nX_n) : \sum_{i=1}^n t_i = t, t_i \in \mathbb{R}^+, \ X_1, \ldots, X_n \in \Omega\}$ and $A_{pc}(e, \Omega)$ is equal to the semigroup generated by the set $\exp \mathbb{R}^+ \Omega$. 

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(2) $A_{pc}(e, t, \Omega)$ is dense in $A(e, t, \hat{\Omega})$.
(3) $A_{pc}(e, \Omega)$ is dense in both $A(e, \Omega)$ and $A(e, \hat{\Omega})$.
(4) If $\Omega$ generates $L(G)$, then $A(e, \hat{\Omega})$ has non-empty interior and is equal to $\text{int}A_{pc}(e, \Omega)$.
(5) If $\Omega$ is compact and convex, then $A(e, T, \Omega)$ and $A(e, T, \Omega)$ are compact.
(6) If $X_1, \cdots, X_n \in \Omega$ form a basis for $L(G)$ and $X = \sum_{i=1}^{n} t_i X_i$, where $t_i > 0$ for each $i = 1, \cdots, n$ and $t = \sum_{i=1}^{n} t_i$, then $\exp X \in \text{int}A_{pc}(e, s, \Omega)$ for each $s > t$.

**DEFINITION.** Let $G$ be a Lie group with its Lie algebra $L(G)$. A non-empty subset $\Omega$ of $L(G)$ is called a non-overlapping control system (abbreviated NOC set) if the corresponding members of the one-parameter semigroup $t \to A(e, t, \Omega)$ are pairwise disjoint.

**LEMMA 1.** Let $\Omega$ be a NOC set of $L(G)$. Then

1. The map $h_{\Omega}$ from $S(\Omega)$ onto $\mathbb{R}^+$, $A(e, t, \Omega) \to t$ is a well-defined homomorphism.
2. $S(\Omega) \cap S(\Omega)^{-1} = \{e\}$. In particular, zero is not contained in $\Omega$.
3. Any two vectors in $\Omega$ are not contained in a one dimensional subspace of $L(G)$.
4. Every subset of NOC set is a NOC set.

**Proof.** (1) If $\Omega$ is a NOC set, then any element attainable for $\Omega$ is only attainable at one time. Thus the map $A(e, t, \Omega) \to t$ is a well-defined homomorphism from $S(\Omega)$ onto $\mathbb{R}^+$. (2) Let $x \in S(\Omega) \cap S(\Omega)^{-1}$. Then $x \in A(e, t, \Omega)$ and $x^{-1} \in A(e, s, \Omega)$ for some $s, t \in \mathbb{R}^+$. Since $e = xx^{-1} \in A(e, t, \Omega)A(e, s, \Omega) = A(e, s+t, \Omega)$, $A(e, 0, \Omega) \cap A(e, s+t, \Omega) \neq \emptyset$. Therefore $0 = s + t$ and hence $s = t = 0$. Note that $A_{pc}(e, 0, \Omega) = \{e\}$ is dense in $A(e, 0, \Omega)$ from Proposition 1. This implies that $A(e, 0, \Omega) = \{e\}$ and hence $x = e$. To prove (3), let $X_0, X_1 \in \Omega$ such that $X_0 = tX_1$ for some $t \neq 1$. If $t > 0, \exp X_0 = \exp tX_1 \in A(e, 1, \Omega) \cap A(e, t, \Omega)$. If $t < 0, \exp X_0 \exp(-t)X_1 = e \in A(e, 0, \Omega) \cap A(e, 1-t, \Omega)$. Since $\Omega$ is a NOC set, which is a contradiction. Finally, since $A(e, t, K) \subset A(e, t, \Omega)$ for any subset $K$ of $\Omega$, it is clear that every subset of NOC set is a NOC set.
Let $\text{Aff}(\mathbb{R})$ denote the unique 2-dimensional non-abelian connected Lie group. It may be identified with the set of ordered pairs $\{(x, y) \in \mathbb{R}^2 \mid x > 0\}$ with multiplication $(a, b)(x, y) = (ax, ay + b)$. Let $\text{aff}(\mathbb{R})$ denote the Lie algebra of $\text{Aff}(\mathbb{R})$. We may identify it with $\{(a, b) \in \mathbb{R}^2 \mid a, b \in \mathbb{R}\}$ under the Lie bracket $[(a, b), (x, y)] = (0, ay - bx)$. A direct calculation yields $\exp A = (e^a, \frac{b}{a}(e^a - 1))$, where $A = (a, b)$ and $\exp$ is a diffeomorphism. Hence $\exp W$ is a closed subsemigroup of $\text{Aff}(\mathbb{R})$ for any wedge $W$ (closed, convex, contains 0, and is additively closed) in $\text{aff}(\mathbb{R})$[3].

**Lemma 2.** In $\text{aff}(\mathbb{R})$, the following conclusions hold:

1. Every non-zero singleton set in $\text{aff}(\mathbb{R})$ is a NOC set.
2. Every non-singleton NOC set generates $\text{aff}(\mathbb{R})$.
3. If $\Omega$ is a non-empty subset of $\text{aff}(\mathbb{R})$ such that $\widehat{\Omega}$ has non-empty interior, then there exist a three vectors $X_0, X_1, X_2 \in \Omega$ such that the closed convex hull of $\{X_0, X_1, X_2\}$ has non-empty interior.

**Proof.** (1) Let $0 \neq X \in \text{aff}(\mathbb{R})$. Then $A(e, t, \{X\}) = \exp tX$. Since $\exp$ is a diffeomorphism, $\{X\}$ is a NOC set. (2) From Lemma 1, every non-singleton NOC set generates $\text{aff}(\mathbb{R})$. (3) Suppose $\widehat{\Omega}$ has non-empty interior. Then we can select a basis $\{X_0, X_1\}$ in $\Omega$. Since $\widehat{\Omega}$ has non-empty interior, we can choose a vector $X_2 \in \Omega$ such that $X_2$ is not contained in the straight line through $X_0$ and $X_1$. Trivially, the closed convex hull of $\{X_0, X_1, X_2\}$ has non-empty interior.

The following Lemma shows that a closed convex hull of NOC set in $\text{aff}(\mathbb{R})$ has empty interior. This implies that a closed convex hull of a NOC set in $\text{aff}(\mathbb{R})$ is contained in a straight line.

**Lemma 3.** Let $\Omega$ be a NOC set of $\text{aff}(\mathbb{R})$. Then $\widehat{\Omega}$ has empty interior.

**Proof.** Suppose that $\widehat{\Omega}$ has non-empty interior. By above note, we may assume that $\Omega = \{X_0, X_1, X_2\}$. Let $W = \mathbb{R}^+X_0 + \mathbb{R}^+X_1 + \mathbb{R}^+X_2$. Then $W \neq \text{aff}(\mathbb{R})$. If not, then $\mathbb{R}^+\widehat{\Omega} = \text{aff}(\mathbb{R})$ and hence $\text{Aff}(\mathbb{R}) = A(e, \widehat{\Omega}) = A(e, \Omega)$ because $A(e, \widehat{\Omega})$ and $A(e, \Omega)$ have the same interior. Therefore we may assume that $W = \mathbb{R}^+X_1 + \mathbb{R}^+X_2$ and $X_0 \in \text{int} W$. 166
Since \( \{X_1, X_2\} \) is a basis for \( \text{aff}(\mathbb{R}) \), \( X_0 = sX_1 + tX_2 \) for some \( s, t \in \mathbb{R}^+ \). If \( s + t = 1 \), then \( \Omega \) has empty interior. Suppose \( s + t < 1 \). Since \( \Omega \) has interior, we find small \( s + t < u < 1 \) such that \( X_0 \in u(\text{int} \Omega) \). Then \( \exp X_0 \in \text{int} \mathbb{A}_{pc}(e, u, \Omega) \subset \mathbb{A}(e, u, \Omega) \) by Proposition 1. This is a contradiction from \( \exp X_0 \in \mathbb{A}(e, 1, \Omega) \). Suppose \( s + t > 1 \). Since \( \mathbb{A}(e, 1, \Omega) \) is dense in \( \mathbb{A}(e, 1, \hat{\Omega}) \), we can choose \( Y \) in \( \text{int} \hat{\Omega} \cap W' \) such that \( \exp Y \in \mathbb{A}(e, 1, \Omega) \), where \( W' = \mathbb{R}^+ X_0 + \mathbb{R}^+ X_1 \). Then \( Y \) lies below of the line segment joining \( X_0 \) and \( X_1 \). Since \( \{X_0, X_1\} \) is a basis for \( \text{aff}(\mathbb{R}) \), \( Y = t' X_0 + s' X_1 \) for some \( s', t' \in \mathbb{R}^+ \) and \( s' + t' < 1 \). Similarly we have a contradiction.

For a non-empty subset \( \Omega \) of \( L(G) \), if \( \Omega \) is interpreted as the set of controls, then it is an important problem to know whether \( \mathbb{A}(e, \Omega) = \mathbb{A}_{pc}(e, \Omega) \)? In the language of control theory, we are asking if piecewise constant bang-bang controls suffice to reach all points in the attainable set. A semigroup formulation is whether each member of \( S(\hat{\Omega}) \) can be written as a finite product of elements from \( \exp(\mathbb{R}^+ \Omega) \). In [4], it is known that if a compact subset \( \Omega \) of \( L(G) \) has the bounded factorization property (that is, there exists \( T > 0 \) and an \( m > 1 \) such that any product \( \exp(t_1 X_1) \cdots \exp(t_{m+1} X_{m+1}) \) with \( \sum_{i=1}^{m+1} t_i \leq T \) and \( X_1, \ldots, X_{m+1} \in \Omega \) can be written as a product \( \exp(s_{1} Y_1) \cdots \exp(s_{m} Y_{m}) \), where \( \sum_{i=1}^{m} s_{i} = \sum_{i=1}^{m+1} t_{i} \) and \( Y_1, \ldots, Y_m \in \Omega \)). Then for each \( t > 0 \), \( \mathbb{A}_{pc}(e, t, \Omega) = \mathbb{A}(e, t, \hat{\Omega}) \) and \( \mathbb{A}_{pc}(e, \Omega) = \mathbb{A}(e, \hat{\Omega}) \).

Let \( W \) be an arbitrary wedge in \( \text{aff}(\mathbb{R}) \) with its bounding rays \( \{tX : t \in \mathbb{R}^+\} \), \( \{tY : t \in \mathbb{R}^+\} \). In [2], I. Chon showed that if \( W \cap I = \{0\} \), where \( I = \{(0,y) : y \in \mathbb{R}\} \) (it is the commutator of \( \text{aff}(\mathbb{R}) \)), then \( \exp W = \exp(\mathbb{R}^+ X) \exp(\mathbb{R}^+ Y) = \exp(\mathbb{R}^+ Y) \exp(\mathbb{R}^+ X) \). Using this result, we have a useful lemma for our approach.

**Lemma 4.** If \( \Omega = \{X, Y\} \) is a NOC set in \( \text{aff}(\mathbb{R}) \) and \( \hat{\Omega} \cap I = \emptyset \), then \( \Omega \) has the bounded factorization property and hence for each \( t > 0 \), \( \mathbb{A}_{pc}(e, t, \Omega) = \mathbb{A}(e, t, \hat{\Omega}) \) and \( \mathbb{A}_{pc}(e, \Omega) = \mathbb{A}(e, \hat{\Omega}) \). Hence \( \hat{\Omega} \) is a NOC set.

**Proof.** Since \( \Omega \) is a NOC set, for any triple product \( \exp tX \exp sY \exp uX \), it is equal to \( \exp tX \exp u'X \exp s'Y = \exp(t + u')X \exp s'Y \), for some \( s', u' \in \mathbb{R}^+ \) such that \( s + u = s' + u' \). Thus \( t + s + u = (t + u') + s' \).
and hence $\Omega$ has the bounded factorization property. This implies that $A(e, t, \hat{\Omega}) = A(e, t, \Omega)$ for any $t > 0$, and hence $\hat{\Omega}$ is a NOC set.

**Lemma 5.** Let $X = (0, a) \in I, Y = (x, y)$ with $x \neq 0$. Then $\exp Y \exp X = \exp(e^y X) \exp Y$.

**Proof.** Straightforward.

**Lemma 6.** Let $X = (x_1, x_2), Y = (y_1, y_2) \in \Omega$. If $x_1 \leq 0$ and $y_1 > 0$ or $x_1 > 0, y_1 \leq 0$, then $\Omega$ is not NOC set.

**Proof.** If $x_1 = 0$, then $X \in \Omega \cap I$. By Lemma 5, $\exp Y \exp X = \exp(e^{y_1} X) \exp Y \in A(e, 2, \Omega) \cap A(e, e^{y_1} + 1, \Omega)$. Since $y_1 > 0$, $\Omega$ is not NOC set. Suppose $x_1 < 0$. Then the closed convex hull of $\{X, Y\}$ meets with $I$ at $Z = (0, a)$. If $a = 0$, then $X$ and $Y$ lie in the one-dimensional subspace of $\text{aff}(\mathbb{R})$ and hence $\Omega$ is not NOC set from Lemma 1. Suppose $\Omega$ is a NOC set. From Lemma 5, $\exp Y \exp Z = \exp(e^{y_1} Z) \exp Y$ and hence $\exp Z, \exp(e^{y_1} Z) \in A(e, t, \Omega)$ for some $t > 0$. However, $\exp(e^{y_1} Z) = \exp Z \exp((0, e^{y_1} a - a))$ and $\exp((0, e^{y_1} a - a)) \in A(e, s, \Omega)$ for some $s > 0$. Therefore $A(e, t, \Omega) \cap A(e, t + s, \Omega) \neq \emptyset$. Since $y_1 > 0$, this is a contradiction.

**Theorem 1.** Let $\Omega$ be a NOC set in $\text{aff}(\mathbb{R})$. Then $\Omega \subset X + I$ for some non-zero vector $X$.

**Proof.** If $\Omega$ does not generate $\text{aff}(\mathbb{R})$, then $\Omega = \{X\}$ for some non-zero vector $X$. Suppose that $\Omega$ generates $\text{aff}(\mathbb{R})$. Since $\hat{\Omega}$ has empty interior, it is a subset of a straight line. If $\hat{\Omega} \cap I \neq \emptyset$, then we can choose $X_1 = (x_1, y_1), X_2 = (x_2, y_2) \in \Omega$ such that $x_1 \leq 0, x_2 > 0$. By Lemma 6, $\{X_1, X_2\}$ is not NOC set. This is a contradiction. Therefore, $\hat{\Omega} \cap I = \emptyset$. Since $\Omega$ generates $\text{aff}(\mathbb{R})$, we have a distinct two vectors $X_1, X_2 \in \Omega$. Then the closed convex hull of $\{X_1, X_2\}$ does not meet with $I$. By Lemma 4, it is a NOC set and hence from the following theorem, it is contained in $X + I$ for some non-zero vector $X$. Therefore $\Omega \subset \hat{\Omega} \subset X + I$ because $\hat{\Omega}$ is a subset of a straight line.

**Theorem 2.** Let $G$ be a connected solvable Lie group. Then a compact, convex generating NOC set must be contained in $X + E$, where $E$ is a subspace of codimension one containing the commutator.

**Proof.** See [1].
Non-overlapping control systems on $Aff(\mathbb{R})$

In [1], we showed that for a bounded subset $\Omega$ of a Lie algebra $L(G)$ of Lie group $G$, $\Omega$ is a NOC set and $h_\Omega$ extends continuously to $G$ if and only if there exists a continuous Lie group homomorphism from $G$ into $\mathbb{R}$ such that its differential map has a non-zero constant value on $\Omega$. From Lemma 1, for any NOC set $\Omega$ in $aff(\mathbb{R})$, it is contained in $X + I$ for some non-zero vector $X \in aff(\mathbb{R})$. Since $I$ is an ideal of $aff(\mathbb{R})$, we have a continuous Lie algebra homomorphism $f$ from $aff(\mathbb{R})$ to $\mathbb{R}$, $tX + Y \rightarrow t$. Since $Aff(\mathbb{R})$ is a simple connected Lie group, we have a continuous Lie group homomorphism from $G$ to $\mathbb{R}$ such that its differential is $f$. Clearly $f(\Omega) = 1$ and hence we conclude that $h_\Omega$ extends continuously to the whole group if $\Omega$ is bounded. More generally, the following conclusion holds:

**Theorem 2.** Let $\Omega$ be a NOC set. Then $h_\Omega$ is continuous and it extends continuously to $Aff(\mathbb{R})$.

**Proof.** The map $f$ from $aff(\mathbb{R})$ to $\mathbb{R}$, $tX + Y \rightarrow t$ is a continuous homomorphism. Hence there exists a continuous Lie group homomorphism $h : Aff(\mathbb{R}) \rightarrow \mathbb{R}$ such that $h \circ \exp = f$, where $\exp$ is the exponential map of Lie group $Aff(\mathbb{R})$. Let $x \in A_{pc}(e, t, \Omega)$. Then $x = \exp t_1 X_1 \cdots \exp t_n X_n$, where $\sum_{i=1}^n t_i = t$ and $X_i \in \Omega$ for $i = 1, 2, \ldots, n$. Since $h(\exp t_i X_i) = t_i, h(x) = t$. Thus $A_{pc}(e, t, \Omega) \subset h^{-1}(t)$. By Proposition 1, $A_{pc}(e, t, \Omega)$ is dense in $A(e, t, \Omega)$ and $h^{-1}(t)$ is closed, we have that $h(A(e, t, \Omega)) = t$, for each $t \in \mathbb{R}^+$. This implies that $h_\Omega = h|_{S(\Omega)}$ extends continuously to $Aff(\mathbb{R})$.

**References**

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