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NON–OVERLAPPING CONTROL SYSTEMS ON $Aff(\mathbf{R})$

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1. Introduction

Let G be a Lie group with Lie algebra $L(G)$ and let Ω be a non-empty subset of $L(G)$. If Ω is interpreted as the set of controls, then the set of elements attainable from the identity for the system Ω is a subsemigroup of G. A system Ω is called a *non–overlapping control system* if any element attainable for Ω is only attainable at one time. In [1], we showed that a compact, convex generating non–overlapping control systems on a connected solvable Lie group must be contained in $X+\mathbf{E}$, where $\mathbf{E}$ is a subspace of codimension one containing the commutator and the homomorphism from the attainable semigroup into $\mathbf{R}^+$ extends continuously to the whole group. In this paper, we show that in $aff(\mathbf{R})$, the unique two dimensional non–abelian Lie algebra, a non–overlapping control system must be contained in $X+[aff(\mathbf{R}), aff(\mathbf{R})]$ and the homomorphism from the attainable semigroup into $\mathbf{R}^+$ extends continuously to $Aff(\mathbf{R})$.

Let G be a Lie group with Lie algebra $L(G)$. Normally we identify $L(G)$ with the set of right invariant vector fields on G. For a non–empty subset Ω of $L(G)$, we consider the control system on G given by the differential equation

$$x'(t) = U(t)(x(t)), \tag{$*$}$$

where U belongs to the class $\mathcal{U}(\Omega)$ of measurable functions from $\mathbf{R}^+ = [0,\infty)$ into Ω which are locally bounded. A solution of $(*)$ is an absolutely continuous function $x(\cdot)$ defined on $\mathbf{R}^+$ such that the equation

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(*) holds almost everywhere. In [8], for $U \in \mathcal{U}(\Omega)$, it is known that there exists a unique solution $x(\cdot)$ on G of the initial value problem

$$x'(t) = U(t)(x(t)), \quad x(0) = g.$$

We denote this solution by $\pi(g, \cdot, U)$, *i.e.*, $\pi(g, 0, U) = g$ and $\pi(g, t, U) = x(t)$ for all $t \in \mathbf{R}^+$. If there exists $U \in \mathcal{U}(\Omega)$ such that $h = \pi(g, t, U)$, then we say that h is *attainable from g at time t for the system Ω*. The set of such elements attainable from g at time t for the system Ω (resp. using only piecewise constant controls into Ω) is denoted $A(g, t, \Omega)$ (resp. $A_{pc}(g, t, \Omega)$). We also employ the notation

$$\mathbf{A}(g, T, \Omega) = \bigcup_{0 \leq t \leq T} A(g, t, \Omega)$$

$$\mathbf{A}(g, \Omega) = \bigcup_{0 \leq t < \infty} A(g, t, \Omega)$$

The set $\mathbf{A}(g, \Omega)$ is called *the attainability set from g*. From the right invariance of the control system, $A(g, T, \Omega) = A(e, T, \Omega)g$, and $\mathbf{A}(g, \Omega) = \mathbf{A}(e, \Omega)g$. Thus we restrict our attention to the attainability set at the identity.

For a non-empty subset Ω of $L(G)$, we have the one-parameter semigroup of sets $t \to A(e, t, \Omega)$. That is, $A(e, s, \Omega)A(e, t, \Omega) = A(e, s + t, \Omega)$ for all $s, t \in \mathbf{R}^+$ ([5],[8]). This implies that the attainability set at the identity is a subsemigroup of G and $S(\Omega) := \mathbf{A}(e, \Omega)$ is called *the attainable semigroup for Ω*. In this paper, we say that Ω *generates* $L(G)$ if $L(G)$ is the smallest subalgebra containing Ω.

2. Non-overlapping control systems on $Aff(\mathbf{R})$

We start by summarizing the useful properties concerning control systems on Lie groups. The following results could be found in [5].

PROPOSITION 1. *Let Ω be a non-empty subset of $L(G)$ and $\widehat{\Omega}$ be its closed convex hull. Then*

(1) $A_{pc}(e, t, \Omega) = \{\exp(t_1 X_1) \cdots \exp(t_n X_n) : \sum_{i=1}^{n} t_i = t, t_i \in \mathbf{R}^+, X_1, \ldots, X_n \in \Omega\}$ *and $\mathbf{A}_{pc}(e, \Omega)$ is equal to the semigroup generated by the set $\exp \mathbf{R}^+ \Omega$.*

(2) $A_{pc}(e,t,\Omega)$ *is dense in* $A(e,t,\widehat{\Omega})$.
(3) $\mathbf{A}_{pc}(e,\Omega)$ *is dense in both* $\mathbf{A}(e,\Omega)$ *and* $\mathbf{A}(e,\widehat{\Omega})$.
(4) *If* Ω *generates* $L(G)$, *then* $\mathbf{A}(e,\widehat{\Omega})$ *has non–empty interior and is equal to* $int\mathbf{A}_{pc}(e,\Omega)$.
(5) *If* Ω *is compact and convex, then* $A(e,T,\Omega)$ *and* $\mathbf{A}(e,T,\Omega)$ *are compact.*
(6) *If* $X_1,\cdots,X_n\in\Omega$ *form a basis for* $L(G)$ *and* $X=\sum_{i=1}^n t_iX_i$, *where* $t_i>0$ *for each* $i=1,\cdots,n$ *and* $t=\sum_{i=1}^n t_i$, *then* $\exp X\in int\mathbf{A}_{pc}(e,s,\Omega)$ *for each* $s>t$.

DEFINITION. Let G be a Lie group with its Lie algebra $L(G)$. A non–empty subset Ω of $L(G)$ is called a *non-overlapping control system* (abbreviated NOC set) if the corresponding members of the one–parameter semigroup $t\rightarrow A(e,t,\Omega)$ are pairwise disjoint.

LEMMA 1. *Let* Ω *be a* NOC *set of* $L(G)$. *Then*

(1) *The map* h_Ω *from* $S(\Omega)$ *onto* $\mathbf{R}^+, A(e,t,\Omega)\rightarrow t$ *is a well–defined homomorphism.*
(2) $S(\Omega)\cap S(\Omega)^{-1}=\{e\}$. *In particular, zero is not contained in* Ω.
(3) *Any two vectors in* Ω *are not contained in a one dimensional subspace of* $L(G)$.
(4) *Every subset of* NOC *set is a* NOC *set.*

Proof. (1) If Ω is a NOC set, then any element attainable for Ω is only attainable at one time. Thus the map $A(e,t,\Omega)\rightarrow t$ is a well–defined homomorphism from $S(\Omega)$ onto $\mathbf{R}^+$. (2) Let $x\in S(\Omega)\cap S(\Omega)^{-1}$. Then $x\in A(e,t,\Omega)$ and $x^{-1}\in A(e,s,\Omega)$ for some $s,t\in\mathbf{R}^+$. Since $e=xx^{-1}\in A(e,t,\Omega)A(e,s,\Omega)=A(e,s+t,\Omega)$, $A(e,0,\Omega)\cap A(e,s+t,\Omega)\neq\phi$. Therefore $0=s+t$ and hence $s=t=0$. Note that $A_{pc}(e,0,\Omega)=\{e\}$ is dense in $A(e,0,\Omega)$ from Proposition 1. This implies that $A(e,0,\Omega)=\{e\}$ and hence $x=e$. To prove (3), let $X_0,X_1\in\Omega$ such that $X_0=tX_1$ for some $t\neq 1$. If $t>0, \exp X_0=\exp tX_1\in A(e,1,\Omega)\cap A(e,t,\Omega)$. If $t<0, \exp X_0\exp(-t)X_1=e\in A(e,0,\Omega)\cap A(e,1-t,\Omega)$. Since Ω is a NOC set, which is a contradiction. Finially, since $A(e,t,K)\subset A(e,t,\Omega)$ for any subset K of Ω, it is clear that every subset of NOC set is a NOC set.

Let $Aff(\mathbf{R})$ denote the unique 2-dimensional non-abelian connected Lie group. It may be identified with the set of ordered pairs $\{(x,y) \in \mathbf{R}^2 \mid x > 0\}$ with multiplication $(a,b)(x,y) = (ax, ay+b)$. Let $aff(\mathbf{R})$ denote the Lie algebra of $Aff(\mathbf{R})$. We may identify it with $\{(a,b) \in \mathbf{R}^2 \mid a, b \in \mathbf{R}\}$ under the Lie bracket $[(a,b),(x,y)] = (0, ay - bx)$. A direct calculation yields $\exp A = (e^a, \frac{b}{a}(e^a - 1))$, where $A = (a,b)$ and exp is a diffeomorphism. Hence $\exp W$ is a closed subsemigroup of $Aff(\mathbf{R})$ for any wedge W (closed, convex, contains 0, and is additively closed) in $aff(\mathbf{R})$[3].

LEMMA 2. *In $aff(\mathbf{R})$, the following conclusions hold:*

(1) *Every non-zero singleton set in $aff(\mathbf{R})$ is a NOC set.*
(2) *Every non-singleton NOC set generates $aff(\mathbf{R})$.*
(3) *If Ω is a non-empty subset of $aff(\mathbf{R})$ such that $\widehat{\Omega}$ has non-empty interior, then there exist a three vectors $X_0, X_1, X_2 \in \Omega$ such that the closed convex hull of $\{X_0, X_1, X_2\}$ has non-empty interior.*

Proof. (1) Let $0 \neq X \in aff(\mathbf{R})$. Then $A(e,t,\{X\}) = \exp tX$. Since exp is a diffeomorphism, $\{X\}$ is a NOC set. (2) From Lemma 1, every non-singleton NOC set generates $aff(\mathbf{R})$. (3) Suppose $\widehat{\Omega}$ has non-empty interior. Then we can select a basis $\{X_0, X_1\}$ in Ω. Since $\widehat{\Omega}$ has non-empty interior, we can choose a vector $X_2 \in \Omega$ such that X_2 is not contained in the straight line through X_0 and X_1. Trivially, the closed convex hull of $\{X_0, X_1, X_2\}$ has non-empty interior.

The following Lemma shows that a closed convex hull of NOC set in $aff(\mathbf{R})$ has empty interior. This implies that a closed convex hull of a NOC set in $aff(\mathbf{R})$ is contained in a straight line.

LEMMA 3. *Let Ω be a NOC set of $aff(\mathbf{R})$. Then $\widehat{\Omega}$ has empty interior.*

Proof. Suppose that $\widehat{\Omega}$ has non-empty interior. By above note, we may assume that $\Omega = \{X_0, X_1, X_2\}$. Let $W = \mathbf{R}^+X_0 + \mathbf{R}^+X_1 + \mathbf{R}^+X_2$. Then $W \neq aff(\mathbf{R})$. If not, then $\mathbf{R}^+\widehat{\Omega} = aff(\mathbf{R})$ and hence $Aff(\mathbf{R}) = \mathbf{A}(e, \widehat{\Omega}) = \mathbf{A}(e, \Omega)$ because $\mathbf{A}(e, \widehat{\Omega})$ and $\mathbf{A}(e, \Omega)$ have the same interior. Therefore we may assume that $W = \mathbf{R}^+X_1 + \mathbf{R}^+X_2$ and $X_0 \in intW$.

Since $\{X_1, X_2\}$ is a basis for $aff(\mathbf{R})$, $X_0 = sX_1 + tX_2$ for some $s, t \in \mathbf{R}^+$. If $s + t = 1$, then $\widehat{\Omega}$ has empty interior. Suppose $s + t < 1$. Since $\widehat{\Omega}$ has interior, we find small $s + t < u < 1$ such that $X_0 \in u(int\widehat{\Omega})$. Then $\exp X_0 \in int\mathbf{A}_{pc}(e, u, \Omega) \subset \mathbf{A}(e, u, \Omega)$ by Proposition 1. This is a contradiction from $\exp X_0 \in A(e, 1, \Omega)$. Suppose $s + t > 1$. Since $A(e, 1, \Omega)$ is dense in $A(e, 1, \widehat{\Omega})$, we can choose Y in $int\widehat{\Omega} \cap W'$ such that $\exp Y \in A(e, 1, \Omega)$, where $W' = \mathbf{R}^+ X_0 + \mathbf{R}^+ X_1$. Then Y lies below of the line segment joining X_0 and X_1. Since $\{X_0, X_1\}$ is a basis for $aff(\mathbf{R})$, $Y = t'X_0 + s'X_1$ for some $s', t' \in \mathbf{R}^+$ and $s' + t' < 1$. Similiary we have a contradition.

For a non-empty subset Ω of $L(G)$, if Ω is interpreted as the set of controls, then it is an important problem to know whether $\mathbf{A}(e, \widehat{\Omega}) = \mathbf{A}_{pc}(e, \Omega)$? In the language of control theory, we are asking if piecewise constant bang–bang controls suffice to reach all points in the attainable set. A semigroup formulation is whether each member of $S(\widehat{\Omega})$ can be written as a finite product of elements from $\exp(\mathbf{R}^+\Omega)$. In [4], it is known that if a compact subset Ω of $L(G)$ has the *bounded factorization* property (that is, there exists $T > 0$ and an $m > 1$ such that any product $\exp(t_1 X_1) \cdots \exp(t_{m+1} X_{m+1})$ with $\sum_{i=1}^{m+1} t_i \leq T$ and $X_1, \cdots, X_{m+1} \in \Omega$ can be written as a product $\exp(s_1 Y_1) \cdots \exp(s_m Y_m)$, where $\sum_{i=1}^{m} s_i = \sum_{i=1}^{m+1} t_i$ and $Y_1, \cdots, Y_m \in \Omega$.), then for each $t > 0$, $A_{pc}(e, t, \Omega) = A(e, t, \widehat{\Omega})$ and $\mathbf{A}_{pc}(e, \Omega) = \mathbf{A}(e, \widehat{\Omega})$.

Let W be an arbitrary wedge in $aff(\mathbf{R})$ with its bounding rays $\{tX : t \in \mathbf{R}^+\}$, $\{tY : t \in \mathbf{R}^+\}$. In [2], I.Chon showed that if $W \cap \mathbf{I} = \{0\}$, where $\mathbf{I} = \{(0, y) : y \in \mathbf{R}\}$(it is the commutator of $aff(\mathbf{R})$), then $\exp W = \exp(\mathbf{R}^+ X)\exp(\mathbf{R}^+ Y) = \exp(\mathbf{R}^+ Y)\exp(\mathbf{R}^+ X)$. Using this result, we have a useful lemma for our approach.

LEMMA 4. *If $\Omega = \{X, Y\}$ is a NOC set in $aff(\mathbf{R})$ and $\widehat{\Omega} \cap I = \phi$, then Ω has the bounded factorization property and hence for each $t > 0$, $A_{pc}(e, t, \Omega) = A(e, t, \widehat{\Omega})$ and $\mathbf{A}_{pc}(e, \Omega) = \mathbf{A}(e, \widehat{\Omega})$. Hence $\widehat{\Omega}$ is a NOC set.*

Proof. Since Ω is a NOC set, for any triple product $\exp tX \exp sY \exp uX$, it is equal to $\exp tX \exp u'X \exp s'Y = \exp(t + u')X \exp s'Y$, for some $s', u' \in \mathbf{R}^+$ such that $s + u = s' + u'$. Thus $t + s + u = (t + u') + s'$

and hence Ω has the bounded factorization property. This implies that $A(e,t,\widehat{\Omega}) = A(e,t,\Omega)$ for any $t > 0$, and hence $\widehat{\Omega}$ is a NOC set.

LEMMA 5. *Let* $X = (0,a) \in I, Y = (x,y)$ *with* $x \neq 0$. *Then* $\exp Y \exp X = \exp(e^x X) \exp Y$.

Proof. Straightforward.

LEMMA 6. *Let* $X = (x_1, x_2), Y = (y_1, y_2) \in \Omega$. *If* $x_1 \leq 0$ *and* $y_1 > 0$ *or* $x_1 > 0$, $y_1 \leq 0$, *then* Ω *is not* NOC *set.*

Proof. If $x_1 = 0$, then $X \in \Omega \cap I$. By Lemma 5, $\exp Y \exp X = \exp(e^{y_1} X) \exp Y \in A(e,2,\Omega) \cap A(e, e^{y_1}+1, \Omega)$. Since $y_1 > 0, \Omega$ is not NOC set. Suppose $x_1 < 0$. Then the closed convex hull of $\{X, Y\}$ meets with I at $Z = (0,a)$. If $a = 0$, then X and Y lie in the one-dimensional subspace of $aff(\mathbf{R})$ and hence Ω is not NOC set from Lemma 1. Suppose Ω is a NOC set. From Lemma 5, $\exp Y \exp Z = \exp(e^{y_1} Z) \exp Y$ and hance $\exp Z, \exp(e^{y_1} Z) \in A(e,t,\Omega)$ for some $t > 0$. However, $\exp(e^{y_1}) Z = \exp Z \exp((0, e^{y_1} a - a))$ and $\exp((0, e^{y_1} a - a)) \in A(e,s,\Omega)$ for some $s > 0$. Therefore $A(e,t,\Omega) \cap A(e,t+s,\Omega) \neq \phi$. Since $y_1 > 0$, this is a contradiction.

THEOREM 1. *Let* Ω *be a* NOC *set in* $aff(\mathbf{R})$. *Then* $\Omega \subset X + \mathbf{I}$ *for some non-zero vector* X.

Proof. If Ω does not generate $aff(\mathbf{R})$, then $\Omega = \{X\}$ for some non-zero vector X. Suppose that Ω generates $aff(\mathbf{R})$. Since $\widehat{\Omega}$ has empty interior, it is a subset of a straight line. If $\widehat{\Omega} \cap \mathbf{I} \neq \phi$, then we can choose $X_1 = (x_1, y_1), X_2 = (x_2, y_2) \in \Omega$ such that $x_1 \leq 0, x_2 > 0$. By Lemma 6, $\{X_1, X_2\}$ is not NOC set. This is a contradiction. Therefore, $\widehat{\Omega} \cap \mathbf{I} = \phi$. Since Ω generates $aff(\mathbf{R})$, we have a distinct two vectors $X_1, X_2 \in \Omega$. Then the closed convex hull of $\{X_1, X_2\}$ does not meet with $\mathbf{I}$. By Lemma 4, it is a NOC set and hence from the following Theorem, it is contained in $X + \mathbf{I}$ for some non-zero vector X. Therefore $\Omega \subset \widehat{\Omega} \subset X + \mathbf{I}$ because $\widehat{\Omega}$ is a subset of a straight line.

THEOREM 2. *Let* G *be a connected solvable Lie group. Then a compact, convex generating* NOC *set must be contained in* $X + \mathbf{E}$, *where* $\mathbf{E}$ *is a subspace of codimension one containing the commutator.*

Proof. See [1].

In [1], we showed that for a bounded subset Ω of a Lie algebra $L(G)$ of Lie group G, Ω is a NOC set and h_Ω extends continuously to G if and only if there exists a continuous Lie group homomorphism from G into $\mathbf{R}$ such that its differential map has a non–zero constant value on Ω. From Lemma 1, for any NOC set Ω in $aff(\mathbf{R})$, it is contained in $X + \mathbf{I}$ for some non–zero vector $X \in aff(\mathbf{R})$. Since $\mathbf{I}$ is an ideal of $aff(\mathbf{R})$, we have a continuous Lie algebra homomorphism f from $aff(\mathbf{R})$ to $\mathbf{R}, tX + Y \to t$. Since $Aff(\mathbf{R})$ is a simple connected Lie group, we have a continuous Lie group homomorphism from G to $\mathbf{R}$ such that its differential is f. Clearly $f(\Omega) = 1$ and hence we conclude that h_Ω extends continuously to the whole group if Ω is bounded. More generally, the following conclusion holds:

THEOREM 2. *Let Ω be a NOC set. Then h_Ω is continuous and it extends continuously to $Aff(\mathbf{R})$.*

Proof. The map f from $aff(\mathbf{R})$ to $\mathbf{R}$, $tX + Y \to t$ is a continuous homomorphism. Hence there exists a continuous Lie group homomorphism $h : Aff(\mathbf{R}) \to \mathbf{R}$ such that $h \circ \exp = f$, where exp is the exponenal map of Lie group $Aff(\mathbf{R})$. Let $x \in A_{pc}(e, t, \Omega)$. Then $x = \exp t_1 X_1 \cdots \exp t_n X_n$, where $\sum_{i=1}^{n} t_i = t$ and $X_i \in \Omega$ for $i = 1, 2, \cdots, n$. Since $h(\exp t_i X_i) = t_i, h(x) = t$. Thus $A_{pc}(e, t, \Omega) \subset h^{-1}(t)$. By Proposition 1, $A_{pc}(e, t, \Omega)$ is dense in $A(e, t, \Omega)$ and $h^{-1}(t)$ is closed, we have that $h(A(e, t, \Omega)) = t$, for each $t \in \mathbf{R}^+$. This implies that $h_\Omega = h|_{S(\Omega)}$ extends continuously to $Aff(\mathbf{R})$.

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