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JORDAN ALGEBRAS ASSOCIATED TO T–ALGEBRAS

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1. Introduction

Let $V \subset \mathbb{R}^n$ be a convex homogeneous cone which does not contain straight lines, so that the automorphism group

$$G = Aut(\mathbb{R}^n, V)^\circ = \{\, g \in GL(\mathbb{R}^n) \mid gV = V\,\}^\circ$$

($\circ$ denoting the identity component) acts transitively on V. A convex cone V is called "self-dual" if V coincides with its dual

$$(1.1) \qquad V' \;= \{\, x' \in \mathbb{R}^n \mid \; < x, x' > \; > \; 0 \;\text{ for all } x \in \overline{V} - \{0\}\,\}$$

where $\overline{V}$ denotes the closure of V.

A Jordan algebra $\mathcal{A}$ over a field F of $char\ F \neq 2$ is a finite dimensional algebra with unite element e such that

(1) $ab = ba$,
(2) $a^2(ba) = (a^2b)a$ for all $a,\ b \in \mathcal{A}$.

A Jordan algebra $\mathcal{A}$ over the field of real numbers $\mathbb{R}$ is said to be *formally real* if the following condition is satisfied

$$(1.2) \qquad x^2 + y^2 = 0 \ (x,\ y \in \mathcal{A}) \;\text{ implies }\; x = y = 0.$$

In 1957-58, M. Koecher made an observation that the category of self-dual convex homogeneous cones $(\mathbb{R}^n, V)$ with a base point $x_0 \in V$ is equivalent to that of formally real Jordan algebras.

By virtue of this equivalence, the classification of self-dual convex homogeneous cones is reduced to that of formally real Jordan algebras, which was given as early as in 1934 ([JNW]). A self-dual convex homogeneous cone V is decomposed uniquely into the direct product of

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the "irreducible" ones. It is well-known that the irreducible self-dual convex homogeneous cones are classified into the following five types :

- $\mathcal{P}_1(\mathbb{R}) = \mathbb{R}_+, \quad \mathcal{P}_r(F) \quad (r \geq 2,\ F = \mathbb{R},\ \mathbb{C},\ \mathbb{H})$,
- $\mathcal{P}_3(\mathbb{O})$ ($\mathbb{O}$ denotes the Cayley octonion algebra),
- $\mathcal{P}(1,\ n-1) = \{ (\xi_i) \in \mathbb{R}^n \mid \xi_1 > 0,\ \xi_1^2 - \sum_{i=2}^{n} \xi_i^2 > 0 \} \quad (n \geq 3)$,

where $\mathcal{P}_r(F)$ denotes the cone of positive definite hermitian matrices of size r with entries in F.

A more general study of "convex homogeneous cones" was done by Vinberg [V] in the early 60's. He showed that there is a one-to-one correspondence between all the convex homogeneous cones and nonassociative algebra of a special form, called compact left-symmetric algebras, or clans and constructed the apparatus of generalized matrix algebras, called T-algebras, which will allow us to consider any convex homogeneous cone as a cone of positive definite hermitian matrices.

The characteristic function plays an essential role in convex cones. For any convex cone V, the characteristic function ϕ_V is defined by

$$\phi_V(x) = \int_{V'} e^{-<x,x'>} dx' \quad \text{for every } x \in V. \tag{1.3}$$

We collect some properties of the characteristic function ϕ_V.

(1) $\phi_V(x) > 0$,
(2) $\phi_V(gx) = det(g)^{-1}\phi_V(x)$ for $x \in V,\ g \in G$,
(3) ϕ_V tends to infinity when $x \in V$ converges to a boundary point of V.
(4) If V_1 and V_2 are open convex cones in the space $\mathbb{R}^n$, $V_1 \cap V_2 \neq \emptyset$, and $\phi_{V_1} = \phi_{V_2}$ on $V_1 \cap V_2$, then $V_1 = V_2$.
(5) The measure $\phi(x)dx$ is invariant under all $g \in G$.
(6) $log\phi_V$ and ϕ_V are convex functions.

We regard the convex cone $V \subset \mathbb{R}^n$ as a differentiable manifold. If x_0 is any point of V and the tangent space to V at x_0 is identified with $\mathbb{R}^n$, the quadratic differential form $d^2 log\phi_V$ provides a Riemannian structure g on V. The components of the Riemannian metric tensor g and the canonical torsion-free connectedness Γ of g are given as follows

:

$$g_{ij} = \partial_{ij} log \phi_V \tag{1.4}$$

$$\Gamma^i_{jk} = \frac{1}{2} \sum g^{il} \partial_{jkl} log \phi_V. \tag{1.5}$$

where (g^{il}) is the tensor inverse of (g_{ij}).

We define a multiplicative operation $\Box$ in $\mathbb{R}^n$ by the formula

$$(a \Box b)^i = -\sum \Gamma^i_{jk}(x_0) a^j b^k \quad (a,\ b \in \mathbb{R}^n). \tag{1.6}$$

This multiplication is commutative, since the connection Γ is torsion free.

The space $\mathbb{R}^n$ with $\Box$ by the formula (1.6) is called the algebra of connectedness of V at x_0.

The purpose of this article is to show that the algebra of connectedness of the convex cone $V(\mathfrak{A})$ corresponding to the T-algebra $\mathfrak{A}$ at the point $e \in V(\mathfrak{A})$ is a formally real Jordan algebra.

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2. Results of T-algebra

In this section, we establish notations and summarize basic facts of the T-algebra.

We consider the square matrices $a = (a_{ij})$ whose elements belong to arbitrary vector spaces :

$$a_{ij} \in \mathfrak{A}_{ij}.$$

A matrix algebra of rank m is an algebra $\mathfrak{A}$ bigraded by subspaces $\mathfrak{A}_{ij}$ $(i,\ j = 1, \cdots, m)$ such that

$$\mathfrak{A}_{ij}\mathfrak{A}_{jk} \subset \mathfrak{A}_{ik}\ ,\quad \mathfrak{A}_{ij}\mathfrak{A}_{lk} = 0 \ \text{ for } j \neq l.$$

We put $\mathfrak{M} = \sum_{i \leq j} \mathfrak{A}_{ij}$. Then the subspace $\mathfrak{M}$ of $\mathfrak{A}$ is a subalgebra of $\mathfrak{A}$. If we are given an involution $a \to a^*$ in the algebra $\mathfrak{A}$, then we can define the subspace

$$\mathfrak{H} = \{\, a \in \mathfrak{A} \mid a^* = a \,\} \tag{2.1}$$

of "hermitian matrices" and the subspace

$$\mathfrak{H}_S = \{ a \in \mathfrak{A} \mid a^* = -a \} \tag{2.2}$$

of "skew-hermitian matrices." Clearly $\mathfrak{A} = \mathfrak{H} + \mathfrak{H}_S$.

NOTATIONS. For all $a,\ b,\ c \in \mathfrak{A}$, we use the following notations :

$$[ab] = ab - ba\ ,\quad [abc] = a(bc) - (ab)c\ ,\quad n_{ij} = \dim\mathfrak{A}_{ij}$$

$$a^\wedge = \frac{1}{2}\sum a_{ii} + \sum_{i<j} a_{ij},\quad a_\vee = \frac{1}{2}\sum a_{ii} + \sum_{i>j} a_{ij}.$$

Note that $a^\wedge$ is an upper triangular matrix and $a_\vee$ is a lower triangular matrix and so $a = a^\wedge + a_\vee$. Also we can easily check that $(a^*)^\wedge = (a_\vee)^*$, $(a^\wedge)^* = (a^*)_\vee$. In particular, for a hermitian matrix a, $a_\vee = (a^\wedge)^*$. In a matrix algebra $\mathfrak{A}$ with involution $*$, $n_{ij} = n_{ji}$ and each subspace $\mathfrak{A}_{ii}$ is a subalgebra isomorphic to the algebra $\mathbb{R}$ of real numbers. The unique isomorphism of $\mathfrak{A}_{ii}$ onto $\mathbb{R}$ will be denoted by ρ and we denote the unit element of $\mathfrak{A}_{ii}$ by e_i.

DEFINITION 2.1. ([V]) A matrix algebra $\mathfrak{A}$ with an involution $*$ is called a T-algebra if the following seven conditions are satisfied :

(1) the subalgbra $\mathfrak{A}_{ii}$ is isomorphic to the algebra $\mathbb{R}$;
(2) for every $a_{ij} \in \mathfrak{A}_{ij}$ $\ e_i a_{ij} = a_{ij} e_j = a_{ij}$;
(3) $tr([ab]) = 0$;
(4) $tr([abc]) = 0$;
(5) $tr(aa^*) > 0$ if $a \neq 0$;
(6) for any $a,\ b,\ c \in \mathfrak{M}$, $[abc] = 0$ or $[a^*b^*c^*] = 0$;
(7) for any $a,\ b,\ c \in \mathfrak{M}$,

$$[abb^*] = 0 \ \text{ or } \ [abc^*] + [acb^*] = 0 \ \text{ or } \ [ab^*c^*] + [ba^*c^*] = 0.$$

We define a bilinear operation $\triangle$ in the space $\mathfrak{A}$ by the formula

$$a \triangle b = a^\wedge b + b a_\vee. \tag{2.3}$$

The space $\mathfrak{H}$ of hermitian matrices is closed under the operation $\triangle$. In fact, for any hermitian matrices $a,\ b \in \mathfrak{H}$,

$$(a \triangle b)^* \ = \ (a^\wedge b + ba_\vee)^* = ba_\vee + a^\wedge b \ = \ a \triangle b.$$

Lemma 2.2.([V]) For any $a \in \mathfrak{H}$, the operator

$$L_a \ : \ b \longrightarrow a \triangle b \quad (b \in \mathfrak{H}) \tag{2.4}$$

in $\mathfrak{H}$ has only real eigenvalues and

$$tr(L_a) = tr(a) = \sum_{i=1}^{m} n_i \rho(a_{ii}) \tag{2.5}$$

where

$$n_i = 1 + \frac{1}{2} \sum_{s \neq i}^{m} n_{is}. \tag{2.6}$$

The axiom (6) in Definition of T-algebra shows that the set of all triangular matrices of $\mathfrak{A}$ forms an associative subalgebra of $\mathfrak{A}$. For any triangular matrix a with $a_{ii} \neq 0$ for all i, a is not a zero divisor in this subalgebra $\mathfrak{M}$. Then it is not hard to prove that

$$T(\mathfrak{A}) = \{\, a \in \mathfrak{M} \mid a_{ii} > 0 \ \ (i = 1, \cdots, m)\,\} \tag{2.7}$$

is open in $\mathfrak{M}$ and is a connected Lie group. Its Lie algebra $\mathfrak{p}(\mathfrak{A})$ can be identified with $\mathfrak{M}$, where the commutator operation [,] is defined by the formula

$$[a,\ b] = [ab].$$

We consider the mapping

$$F \ : \ a \longrightarrow aa^* \in \mathfrak{H} \quad (a \in \mathfrak{M}). \tag{2.8}$$

Let e be the identity of $\mathfrak{M}$. Then the differential mapping dF_e is of the form

$$dF_e \ : \ a \longrightarrow a + a^*$$

and is an isomorphism of the linear space $\mathfrak{M}$ onto $\mathfrak{H}$. Therefore the image of $T(\mathfrak{A})$ under F contains the matrix $F(e) = e$ in $\mathfrak{H}$.

Let

$$V(\mathfrak{A}) = F(T(\mathfrak{A})) = \{\, tt^* \mid t \in T(\mathfrak{A})\,\} \tag{2.9}$$

THEOREM 2.3. ([V], Theorem 4, p. 397)

$$\{\, convex\ homogeneous\ cones \,\} \ \cong\ \{\, T-algebras \,\}.$$

$$V \cong V(\mathfrak{A}) \qquad \longleftrightarrow \qquad \mathfrak{A}$$

In the T-algebra $\mathfrak{A}$ of rank m we consider the subspace

$$\mathfrak{A}^{(k)} = \sum_{i,j=1}^{k} \mathfrak{A}_{ij} \quad (k = 1, \cdots, m). \tag{2.10}$$

For every hermitian matrix $a \in \mathfrak{H}$, we construct a sequence of matrices $a^{(k)} = (a_{ij}^{(k)}) \in \mathfrak{A}^{(k)}$ $(k = 1, \cdots, m)$ as follows :

$$a^{(m)} = a \ , \quad a^{(k-1)} = \sum_{i,j=1}^{k-1} (\rho(a_{kk}^{(k)})a_{ij}^{(k)} - a_{ik}^{(k)} a_{kj}^{(k)}).$$

Let

$$p_k(a) = \rho(a_{kk}^{(k)}) \ \ (k = 1, \cdots, m) \tag{2.11}$$

It is easy to see that $p_k(a)$ is a homogeneous polynomial of degree 2^{m-k} in the coordinates of the vector $a \in \mathfrak{H}$.

We collect some properties of the polynomial $p_k(a)$.

(1) *If $a = bb^*$, where $b \in \mathfrak{M}$, then*

$$a_{ij}^{(k)} = (\prod_{s>k} p_s(a)) \sum_{l=1}^{k} b_{il}b_{jl}^*.$$

(2) *The cone $V(\mathfrak{A})$ is determined in $\mathfrak{H}$ by the inequalities*

$$p_k(a) > 0 \ \ (k = 1, \cdots, m),$$

and every hermitian matrix $a \in V(\mathfrak{A})$ can be written in exactly one way in the form tt^*, where $t \in T(\mathfrak{A})$.

(3) *If ϕ is a characteristic function on $V(\mathfrak{A})$, then*

$$\phi(a) = \prod_{i=1}^{m} (p_i(a))^{n_1+n_2+\cdots+n_{i-1}-n_i}.$$

3. Main Theorem

In Introduction we get the value of the metric tensor g and the object of connectedness at the base point x_0 in the convex cone V. Now, we apply these results to the convex cone $V(\mathfrak{A})$ corresponding to the T-algebra $\mathfrak{A}$. In this case the role of x_0 will be taken by the unit matrix e.

Let $\mathfrak{A}$ be a T-algebra with operation $\triangle$ by (2.3) and $V(\mathfrak{A})$ the convex cone corresponding to the T-algebra $\mathfrak{A}$. Let ϕ be the characteristic function of $V(\mathfrak{A})$. We may assume that $\phi(e) = 1$. For any $a \in \mathfrak{H}$, we can use the operator L_a given by the formula (2.4) in Lemma 2.2.([V]) in the following facts.

Since

$$\phi((expL_a)e) = (det\ expL_a)^{-1}\phi(e) = e^{-tr(L_a)},$$

we get

$$log\phi((expL_a)e) = -tr(L_a).$$

Note that

$$\begin{aligned}(expL_a)e &= e + \sum_{k=0}^{\infty} \frac{L_a^k}{(k+1)!} a \\ &= e + a + \frac{1}{2}L_a a + \frac{1}{6}L_a L_a a + \cdots \\ &= e + a + \frac{1}{2}a\triangle a + \frac{1}{6}a\triangle(a\triangle a) + \cdots \quad .\end{aligned}$$

We calculate the first few terms in the Taylor series expansion of $log\phi$ in the neighborhood at $e \in V(\mathfrak{A})$ as follows :

$$\begin{aligned}-tr(L_a) =& log\phi((expL_a)e) \\ =& (d\ log\phi(e))(e + a + \frac{1}{2}a\triangle a + \frac{1}{6}a\triangle(a\triangle a)) \\ &+ \frac{1}{2}(d^2 log\phi(e))(a + \frac{1}{2}a\triangle a) + \frac{1}{6}(d^3 log\phi(e))(a) + \cdots \quad .\end{aligned}$$

Let $g(e)$ be the symmetric bilinear form connected with the quadratic form $d^2 log\phi(e)$. Then

$$\begin{aligned}(d^2 log\phi(e))(a+\frac{1}{2}a\triangle a) =& g(e)(a+\frac{1}{2}a\triangle a, a+\frac{1}{2}a\triangle a)\\ =& g(e)(a,a)+g(e)(a,a\triangle a)\\ &+\frac{1}{4}g(e)(a\triangle a, a\triangle a)\\ =&(d^2 log\phi(e))(a)+\frac{1}{4}(d^2 log\phi(e))(a\triangle a)\\ &+g(e)(a,a\triangle a).\end{aligned}$$

Thus

$$\begin{aligned}&-tr(L_a)\\ &\quad=(d\ log\phi(e))(a)+\frac{1}{2}[(d\ log\phi(e))(a\triangle a)+(d^2 log\phi(e))(a)]\\ &\quad+\frac{1}{6}[(d\ log\phi(e))(a\triangle(a\triangle a))+3g(e)(a,a\triangle a)+(d^3 log\phi(e))(a)]\\ &\quad+\cdots\ .\end{aligned}$$

By comparing terms of the first order, the second order and the third order of smallness and Lemma 2.2.([V]), we obtain

(3.1) $$(d\ log\phi(e))(a) = -tr(L_a) = -tr(a)\ ,$$

(3.2) $$(d^2 log\phi(e))(a) = -(d\ log\phi(e))(a\triangle a) = tr(L_{a\triangle a}) = tr(a\triangle a)$$

and

(3.3) $$(d^3 log\phi(e))(a) = -2tr(a\triangle(a\triangle a))\ .$$

Since $tr(a\triangle b) = tr(a^\wedge b\ +\ ba_\vee) = tr((a^\wedge + a_\vee)b) = tr(ab)$, it follows that

$$g(e)(a,b) = tr(ab).$$

The algebra of connectedness of the cone $V(\mathfrak{A})$ at e is defined by the multiplicative operation $\square$ in $\mathfrak{H}$ as follows :

$$(a \square b)^i = -\sum \Gamma^i_{jk}(e) a^j b^k \ , \quad a,\ b \in \mathfrak{H}$$

where

$$\Gamma^i_{jk}(e) = \frac{1}{2} \sum g^{il}(e) \partial_{jkl} log\phi(e).$$

Thus we have

$$\begin{aligned} g(e)(a\square b, c) &= \sum g_{il}(e)(a\square b)^i c^l. \\ &= -\sum g_{il}(e)\Gamma^i_{jk}(e) a^j b^k c^l. \\ &= -\frac{1}{2}\sum g_{il}(e) g^{it}(e)\partial_{jkt} log\phi(e) a^j b^k c^l. \\ &= -\frac{1}{2}\partial_{jkl} log\phi(e) a^j b^k c^l \end{aligned}$$

Let $Q(a,b,c) = \partial_{jkl} log\phi(e) a^j b^k c^l$. Then Q is symmetric trilinear and so

$$g(e)(a\square b, c) = -\frac{1}{2} Q(a,b,c) \ ,$$

by (3.3), we get

$$Q(a,a,a) = (d^3 log\phi(e))(a) = -2tr(a\triangle(a\triangle a)).$$

LEMMA 3.1. *Let $R(a,b,c)$ be the trilinear form and let $Q(a,b,c)$ be the symmetric trilinear form. If $R(a,b,c)$ is symmetric and $R(a,a,a) = Q(a,a,a)$, then $R(a,b,c) = Q(a,b,c)$ for all $a,\ b,\ c \in \mathfrak{H}$.*

Proof. It follows immediately from the polarization. □

MAIN THEOREM. *Let $\mathfrak{A}$ be a T-algebra and $V(\mathfrak{A})$ the corresponding convex homogeneous cone in the space $\mathfrak{H}$ of hermitian matrices. Then the algebra of connectedness of the cone $V(\mathfrak{A})$ at the point $e \in V(\mathfrak{A})$ is a formally real Jordan algebra.*

Proof. Let $R(a,b,c) = g(e)(ab+ba, c)$. Then $R(a,b,c)$ is symmetric because

$$R(a,b,c) = R(b,a,c)$$

and

$$\begin{aligned} R(a,c,b) &= g(e)(ac+ca,b) \\ &= tr((ac)b+(ca)b) = tr((ab+ba)c) \quad \textit{(by axiom (4))} \\ &= g(e)(ab+ba,c) = R(a,b,c)\ . \end{aligned}$$

Moreover,

$$\begin{aligned} tr(a\triangle(a\triangle a)) &= tr(a(a\triangle a)) = tr(a(a^{\wedge}a + a_{\vee}a)) \\ &= tr(a^2(a^{\wedge}+a_{\vee})) = tr((a^2)a) \\ &= g(e)(a^2,a) = \frac{1}{2}R(a,a,a) \end{aligned}$$

Thus we get $R(a,a,a) = -Q(a,a,a)$ *and for all* $a,\ b,\ c \in \mathfrak{H}$, *we have*

$$g(e)(ab+ba,c) = R(a,b,c) = -Q(a,b,c) = 2g(e)(a\Box b,c)$$

by Lemma 3.1. Hence

$$a\Box b = \frac{1}{2}(ab+ba).$$

Since the algebra $(\mathfrak{H},\Box)$ *is commutative, it suffices to show that* $(\mathfrak{H},\Box)$ *satisfies the Jordan identity and formally real. If* $a,\ b,\ c \in \mathfrak{H}$,

$$\begin{aligned} &tr((a^2\Box(b\Box a))c) \\ &= \frac{1}{4}\{tr(a^2(ba)) + tr(a^2(ab)) + tr((ba)a^2) + tr((ab)a^2)\}c \\ &= \frac{1}{4}\{tr((a^2b)a) + tr((ba^2)a) + tr(a(a^2b)) + tr(a(ba^2))\}c \\ &= tr(((a^2\Box b)\Box a)c). \end{aligned}$$

by axiom (3) and (4). Thus if $a,\ b \in \mathfrak{H}$, *we have*

$$g(e)(a^2\Box(b\Box a),c) = g(e)((a^2\Box b)\Box a,c) \quad \textit{for all } c \in \mathfrak{H},$$

and so $a^2\Box(b\Box a) = (a^2\Box b)\Box a$.
Suppose that $a\Box a + b\Box b = 0$. *Since* $a\Box a + b\Box b = a^2+b^2 = 0$,

$$\begin{aligned} g(e)(a,a) + g(e)(b,b) &= tr(a^2) + tr(b^2) = tr(a^2+b^2) \\ &= g(e)(a^2+b^2,e) = 0\ . \end{aligned}$$

By axiom (5), if $a \neq 0$, *then* $tr(a^2) > 0$, *and hence* $a = b = 0$.
This completes the proofs. □

References

[JNW] P. Jordan, J. von Neumann and E. Wigner, *On an algebraic generalization of the quantum mechanical formalism*, Ann. of Math. **35** (1934), 29-64.

[V] E. B. Vinberg, *The theory of convex homogeneous cones*, Trans. Moscow Math. Soc. (1963), 340-403.

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