JORDAN ALGEBRAS ASSOCIATED TO T-ALGEBRAS

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1. Introduction

Let \( V \subset \mathbb{R}^n \) be a convex homogeneous cone which does not contain straight lines, so that the automorphism group

\[
G = \text{Aut}(\mathbb{R}^n, V)^\circ = \{ g \in GL(\mathbb{R}^n) \mid gV = V \}^\circ
\]

(\( ^\circ \) denoting the identity component) acts transitively on \( V \). A convex cone \( V \) is called "self-dual" if \( V \) coincides with its dual

\[
(1.1) \quad V' = \{ x' \in \mathbb{R}^n \mid \langle x, x' \rangle > 0 \text{ for all } x \in V - \{0\} \}
\]

where \( V \) denotes the closure of \( V \).

A Jordan algebra \( \mathcal{A} \) over a field \( F \) of \( \text{char} \ F \neq 2 \) is a finite dimensional algebra with unit element \( e \) such that

1. \( ab = ba \),
2. \( a^2(ba) = (a^2b)a \) for all \( a, b \in \mathcal{A} \).

A Jordan algebra \( \mathcal{A} \) over the field of real numbers \( \mathbb{R} \) is said to be \textit{formally real} if the following condition is satisfied

\[
(1.2) \quad x^2 + y^2 = 0 \ (x, y \in \mathcal{A}) \text{ implies } x = y = 0.
\]

In 1957-58, M. Koecher made an observation that the category of self-dual convex homogeneous cones \( (\mathbb{R}^n, V) \) with a base point \( x_0 \in V \) is equivalent to that of formally real Jordan algebras.

By virtue of this equivalence, the classification of self-dual convex homogeneous cones is reduced to that of formally real Jordan algebras, which was given as early as in 1934 ([JNW]). A self-dual convex homogeneous cone \( V \) is decomposed uniquely into the direct product of

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the "irreducible" ones. It is well-known that the irreducible self-dual convex homogeneous cones are classified into the following five types:

- \( \mathcal{P}_1(\mathbb{R}) = \mathbb{R}_+ \), \( \mathcal{P}_r(F) \quad (r \geq 2, \ F = \mathbb{R}, \mathbb{C}, \mathbb{H}) \),
- \( \mathcal{P}_3(\mathbb{O}) \) (\( \mathbb{O} \) denotes the Cayley octonion algebra),
- \( \mathcal{P}(1, n-1) = \{ (\xi_i) \in \mathbb{R}^n \mid \xi_1 > 0, \xi_1^2 - \sum_{i=2}^{n} \xi_i^2 > 0 \} \) (\( n \geq 3 \)),

where \( \mathcal{P}_r(F) \) denotes the cone of positive definite hermitian matrices of size \( r \) with entries in \( F \).

A more general study of "convex homogeneous cones" was done by Vinberg [V] in the early 60's. He showed that there is a one-to-one correspondence between all the convex homogeneous cones and nonassociative algebra of a special form, called compact left-symmetric algebras, or clans and constructed the apparatus of generalized matrix algebras, called T-algebras, which will allow us to consider any convex homogeneous cone as a cone of positive definite hermitian matrices.

The characteristic function plays an essential role in convex cones. For any convex cone \( V \), the characteristic function \( \phi_V \) is defined by

\[
(1.3) \quad \phi_V(x) = \int_{V'} e^{-<x, x'>} dx' \quad \text{for every } x \in V.
\]

We collect some properties of the characteristic function \( \phi_V \).

1. \( \phi_V(x) > 0 \),
2. \( \phi_V(gx) = det(g)^{-1} \phi_V(x) \) for \( x \in V \), \( g \in G \),
3. \( \phi_V \) tends to infinity when \( x \in V \) converges to a boundary point of \( V \).
4. If \( V_1 \) and \( V_2 \) are open convex cones in the space \( \mathbb{R}^n \), \( V_1 \cap V_2 \neq \emptyset \), and \( \phi_{V_1} = \phi_{V_2} \) on \( V_1 \cap V_2 \), then \( V_1 = V_2 \).
5. The measure \( \phi(x)dx \) is invariant under all \( g \in G \).
6. \( \log \phi_V \) and \( \phi_V \) are convex functions.

We regard the convex cone \( V \subset \mathbb{R}^n \) as a differentiable manifold. If \( x_0 \) is any point of \( V \) and the tangent space to \( V \) at \( x_0 \) is identified with \( \mathbb{R}^n \), the quadratic differential form \( d^2 \log \phi_V \) provides a Riemannian structure \( g \) on \( V \). The components of the Riemannian metric tensor \( g \) and the canonical torsion-free connectedness \( \Gamma \) of \( g \) are given as follows
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\[(1.4) \quad g_{ij} = \partial_{ij} \log \phi_V \]
\[(1.5) \quad \Gamma^i_{jk} = \frac{1}{2} \sum g^{il} \partial_{jk} \log \phi_V. \]

where \((g^{ij})\) is the tensor inverse of \((g_{ij})\).

We define a multiplicative operation \( \Box \) in \( \mathbb{R}^n \) by the formula

\[(1.6) \quad (a \Box b)^i = -\sum \Gamma^i_{jk}(x_0) a^j b^k \quad (a, \ b \in \mathbb{R}^n). \]

This multiplication is commutative, since the connection \( \Gamma \) is torsion free.

The space \( \mathbb{R}^n \) with \( \Box \) by the formula (1.6) is called the algebra of connectedness of \( V \) at \( x_0 \).

The purpose of this article is to show that the algebra of connectedness of the convex cone \( V(\mathfrak{A}) \) corresponding to the \( T \)-algebra \( \mathfrak{A} \) at the point \( e \in V(\mathfrak{A}) \) is a formally real Jordan algebra.

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2. Results of \( T \)-algebra

In this section, we establish notations and summarize basic facts of the \( T \)-algebra.

We consider the square matrices \( a = (a_{ij}) \) whose elements belong to arbitrary vector spaces :

\[a_{ij} \in \mathfrak{A}_{ij}.\]

A matrix algebra of rank \( m \) is an algebra \( \mathfrak{A} \) bigraded by subspaces \( \mathfrak{A}_{ij} \ (i, j = 1, \cdots, m) \) such that

\[\mathfrak{A}_{ij} \mathfrak{A}_{jk} \subset \mathfrak{A}_{ik}, \quad \mathfrak{A}_{ij} \mathfrak{A}_{lk} = 0 \quad \text{for} \ j \neq l.\]

We put \( \mathfrak{M} = \sum_{i \leq j} \mathfrak{A}_{ij} \). Then the subspace \( \mathfrak{M} \) of \( \mathfrak{A} \) is a subalgebra of \( \mathfrak{A} \). If we are given an involution \( a \to a^* \) in the algebra \( \mathfrak{A} \), then we can define the subspace

\[(2.1) \quad \mathfrak{H} = \{ a \in \mathfrak{A} \mid a^* = a \} \]

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of "hermitian matrices" and the subspace

\[(2.2) \quad \mathfrak{H}_S = \{ a \in \mathfrak{A} \mid a^* = -a \}\]

of "skew-hermitian matrices." Clearly \( \mathfrak{A} = \mathfrak{H} + \mathfrak{H}_S \).

**NOTATIONS.** For all \( a, b, c \in \mathfrak{A} \), we use the following notations:

\[ [ab] = ab - ba, \quad [abc] = a(bc) - (ab)c, \quad n_{ij} = \dim \mathfrak{A}_{ij} \]

\[ a^\wedge = \frac{1}{2} \sum a_{ii} + \sum_{i<j} a_{ij}, \quad a_\vee = \frac{1}{2} \sum a_{ii} + \sum_{i>j} a_{ij}. \]

Note that \( a^\wedge \) is an upper triangular matrix and \( a_\vee \) is a lower triangular matrix and so \( a = a^\wedge + a_\vee \). Also we can easily check that \( (a^*)^\wedge = (a_\vee)^*, \quad (a^*)^\vee = (a^*)_\wedge \). In particular, for a hermitian matrix \( a \), \( a_\vee = (a^\wedge)^* \). In a matrix algebra \( \mathfrak{A} \) with involution \( * \), \( n_{ij} = n_{ji} \) and each subspace \( \mathfrak{A}_{ii} \) is a subalgebra isomorphic to the algebra \( \mathbb{R} \) of real numbers. The unique isomorphism of \( \mathfrak{A}_{ii} \) onto \( \mathbb{R} \) will be denoted by \( \rho \) and we denote the unit element of \( \mathfrak{A}_{ii} \) by \( e_i \).

**DEFINITION 2.1.** ([V]) A matrix algebra \( \mathfrak{A} \) with an involution \( * \) is called a T-algebra if the following seven conditions are satisfied:

1. the subalgebra \( \mathfrak{A}_{ii} \) is isomorphic to the algebra \( \mathbb{R} \);
2. for every \( a_{ij} \in \mathfrak{A}_{ij} \), \( e_i a_{ij} = a_{ij} e_j = a_{ij} \);
3. \( tr([ab]) = 0 \); 
4. \( tr([abc]) = 0 \);
5. \( tr(aa^*) > 0 \) if \( a \neq 0 \);
6. for any \( a, b, c \in \mathfrak{M} \), \( [abc] = 0 \) or \( [a^*b^*c^*] = 0 \);
7. for any \( a, b, c \in \mathfrak{M} \),

\[ [abb^*] = 0 \quad \text{or} \quad [abc^*] + [acb^*] = 0 \quad \text{or} \quad [ab^*c^*] + [ba^*c^*] = 0. \]

We define a bilinear operation \( \Delta \) in the space \( \mathfrak{A} \) by the formula

\[(2.3) \quad a \Delta b = a^\wedge b + ba_\vee.\]
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The space \( \mathfrak{H} \) of hermitian matrices is closed under the operation \( \triangle \). In fact, for any hermitian matrices \( a, b \in \mathfrak{H} \),

\[
(a \triangle b)^* = (a^\wedge b + ba_\vee)^* = ba_\vee + a^\wedge b = a \triangle b.
\]

**Lemma 2.2.** ([V]) For any \( a \in \mathfrak{H} \), the operator

\[
L_a : b \rightarrow a \triangle b \quad (b \in \mathfrak{H})
\]

in \( \mathfrak{H} \) has only real eigenvalues and

\[
tr(L_a) = tr(a) = \sum_{i=1}^{m} n_i \rho(a_{ii})
\]

where

\[
n_i = 1 + \frac{1}{2} \sum_{s \neq i} n_{is}.
\]

The axiom (6) in Definition of T-algebra shows that the set of all triangular matrices of \( \mathfrak{A} \) forms an associative subalgebra of \( \mathfrak{A} \). For any triangular matrix \( a \) with \( a_{ii} \neq 0 \) for all \( i \), \( a \) is not a zero divisor in this subalgebra \( \mathfrak{M} \). Then it is not hard to prove that

\[
T(\mathfrak{A}) = \{ a \in \mathfrak{M} \mid a_{ii} > 0 \quad (i = 1, \ldots, m) \}
\]

is open in \( \mathfrak{M} \) and is a connected Lie group. Its Lie algebra \( \mathfrak{p}(\mathfrak{A}) \) can be identified with \( \mathfrak{M} \), where the commutator operation \( [\ , \ ] \) is defined by the formula

\[
[a, b] = [ab].
\]

We consider the mapping

\[
F : a \rightarrow aa^* \in \mathfrak{H} \quad (a \in \mathfrak{M}).
\]

Let \( \epsilon \) be the identity of \( \mathfrak{M} \). Then the differential mapping \( dF_\epsilon \) is of the form

\[
dF_\epsilon : a \rightarrow a + a^*
\]

and is an isomorphism of the linear space \( \mathfrak{M} \) onto \( \mathfrak{H} \). Therefore the image of \( T(\mathfrak{A}) \) under \( F \) contains the matrix \( F(\epsilon) = \epsilon \) in \( \mathfrak{H} \).

Let

\[
V(\mathfrak{A}) = F(T(\mathfrak{A})) = \{ tt^* \mid t \in T(\mathfrak{A}) \}
\]

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THEOREM 2.3. ([V], Theorem 4, p. 397)

\{ convex homogeneous cones \} \cong \{ T - algebras \}.

\[ V \cong V(\mathfrak{A}) \quad \leftrightarrow \quad \mathfrak{A} \]

In the \( T \)-algebra \( \mathfrak{A} \) of rank \( m \) we consider the subspace

\[ \mathfrak{A}^{(k)} = \sum_{i,j=1}^{k} \mathfrak{A}_{ij} \quad (k = 1, \cdots, m). \]

For every hermitian matrix \( a \in \mathfrak{H} \), we construct a sequence of matrices
\[ a^{(k)} = (a^{(k)}_{ij}) \in \mathfrak{A}^{(k)} \quad (k = 1, \cdots, m) \]
as follows:

\[ a^{(m)} = a, \quad a^{(k-1)} = \sum_{i,j=1}^{k-1} (\rho(a_{kk}^{(k)}a_{ij}^{(k)} - a_{ik}^{(k)}a_{kj}^{(k)}). \]

Let

\[ p_k(a) = \rho(a_{kk}^{(k)}) \quad (k = 1, \cdots, m) \]

It is easy to see that \( p_k(a) \) is a homogeneous polynomial of degree \( 2^{m-k} \) in the coordinates of the vector \( a \in \mathfrak{H} \).

We collect some properties of the polynomial \( p_k(a) \).

1. If \( a = bb^* \), where \( b \in \mathfrak{M} \), then

\[ a_{ij}^{(k)} = (\prod_{s > k} p_s(a)) \sum_{l=1}^{k} b_{il}b_{jl}^*. \]

2. The cone \( V(\mathfrak{A}) \) is determined in \( \mathfrak{H} \) by the inequalities

\[ p_k(a) > 0 \quad (k = 1, \cdots, m), \]

and every hermitian matrix \( a \in V(\mathfrak{A}) \) can be written in exactly one way in the form \( tt^* \), where \( t \in T(\mathfrak{A}) \).

3. If \( \phi \) is a characteristic function on \( V(\mathfrak{A}) \), then

\[ \phi(a) = \prod_{i=1}^{m} (p_i(a))^{n_1 + n_2 + \cdots + n_{i-1} - n_i}. \]
3. Main Theorem

In Introduction we get the value of the metric tensor $g$ and the object of connectedness at the base point $x_0$ in the convex cone $V$. Now, we apply these results to the convex cone $V(\mathfrak{A})$ corresponding to the T-algebra $\mathfrak{A}$. In this case the role of $x_0$ will be taken by the unit matrix $e$.

Let $\mathfrak{A}$ be a T-algebra with operation $\triangle$ by (2.3) and $V(\mathfrak{A})$ the convex cone corresponding to the T-algebra $\mathfrak{A}$. Let $\phi$ be the characteristic function of $V(\mathfrak{A})$. We may assume that $\phi(e) = 1$. For any $a \in \mathfrak{H}$, we can use the operator $L_a$ given by the formula (2.4) in Lemma 2.2.([V]) in the following facts.

Since

$$\phi((\exp L_a)e) = (\det \exp L_a)^{-1} \phi(e) = e^{-\text{tr}(L_a)},$$

we get

$$\log \phi((\exp L_a)e) = -\text{tr}(L_a).$$

Note that

$$(\exp L_a)e = e + \sum_{k=0}^{\infty} \frac{L_a^k}{(k+1)!}a$$

$$= e + a + \frac{1}{2}L_a a + \frac{1}{6}L_a L_a a + \cdots$$

$$= e + a + \frac{1}{2}a \triangle a + \frac{1}{6}a \triangle (a \triangle a) + \cdots.$$

We calculate the first few terms in the Taylor series expansion of $\log \phi$ in the neighborhood at $e \in V(\mathfrak{A})$ as follows:

$$-\text{tr}(L_a) = \log \phi((\exp L_a)e)$$

$$= (d \log \phi(e))(e + a + \frac{1}{2} a \triangle a + \frac{1}{6} a \triangle (a \triangle a))$$

$$+ \frac{1}{2} (d^2 \log \phi(e))(a + \frac{1}{2} a \triangle a) + \frac{1}{6} (d^3 \log \phi(e))(a) + \cdots.$$
Let \( g(e) \) be the symmetric bilinear form connected with the quadratic form \( d^2 \log \phi(e) \). Then

\[
(d^2 \log \phi(e))(a + \frac{1}{2} a \Delta a) = g(e)(a + \frac{1}{2} a \Delta a, a + \frac{1}{2} a \Delta a) \\
= g(e)(a, a) + g(e)(a, a \Delta a) \\
+ \frac{1}{4} g(e)(a \Delta a, a \Delta a) \\
= (d^2 \log \phi(e))(a) + \frac{1}{4} (d^2 \log \phi(e))(a \Delta a) \\
+ g(e)(a, a \Delta a).
\]

Thus

\[
-tr(L_a) \\
= (d \log \phi(e))(a) + \frac{1}{2} [(d \log \phi(e))(a \Delta a) + (d^2 \log \phi(e))(a)] \\
+ \frac{1}{6} [(d \log \phi(e))(a \Delta (a \Delta a)) + 3g(e)(a, a \Delta a) + (d^3 \log \phi(e))(a)] \\
+ \cdots.
\]

By comparing terms of the first order, the second order and the third order of smallness and Lemma 2.2.([V]), we obtain

(3.1) \hspace{1cm} (d \log \phi(e))(a) = -tr(L_a) = -tr(a),

(3.2) \hspace{1cm} (d^2 \log \phi(e))(a) = -(d \log \phi(e))(a \Delta a) = tr(L_{a \Delta a}) = tr(a \Delta a)

and

(3.3) \hspace{1cm} (d^3 \log \phi(e))(a) = -2tr(a \Delta (a \Delta a)).

Since \( tr(a \Delta b) = tr(a \Delta + b \Delta a) = tr((a \Delta + a \Delta) b) = tr(ab) \), it follows that

\[ g(e)(a, b) = tr(ab). \]
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The algebra of connectedness of the cone \( V(\mathfrak{A}) \) at \( e \) is defined by the multiplicative operation \( \Box \) in \( \mathfrak{H} \) as follows:

\[
(a \Box b)^i = -\sum \Gamma_{jk}^i(e) a^j b^k, \quad a, b \in \mathfrak{H}
\]

where

\[
\Gamma_{jk}^i(e) = \frac{1}{2} \sum g_{il}^j(e) \partial_{jkli} \log \phi(e).
\]

Thus we have

\[
g(e)(a \Box b, c) = \sum g_{il}^j(e)(a \Box b)^i c^l.
\]

\[
= -\sum g_{il}^j(e) \Gamma_{jk}^i(e) a^j b^k c^l.
\]

\[
= -\frac{1}{2} \sum g_{il}^j(e) g_{jl}^i(e) \partial_{jkli} \log \phi(e) a^j b^k c^l.
\]

\[
= -\frac{1}{2} \partial_{jkli} \log \phi(e) a^j b^k c^l
\]

Let \( Q(a, b, c) = \partial_{jkli} \log \phi(e) a^j b^k c^l \). Then \( Q \) is symmetric trilinear and so

\[
g(e)(a \Box b, c) = -\frac{1}{2} Q(a, b, c),
\]

by (3.3), we get

\[
Q(a, a, a) = (d^3 \log \phi(e))(a) = -2 \text{tr}(a \Delta (a \Delta a)).
\]

**Lemma 3.1.** Let \( R(a, b, c) \) be the trilinear form and let \( Q(a, b, c) \) be the symmetric trilinear form. If \( R(a, b, c) \) is symmetric and \( R(a, a, a) = Q(a, a, a) \), then \( R(a, b, c) = Q(a, b, c) \) for all \( a, b, c \in \mathfrak{H} \).

**Proof.** It follows immediately from the polarization. \( \square \)

**Main Theorem.** Let \( \mathfrak{A} \) be a T-algebra and \( V(\mathfrak{A}) \) the corresponding convex homogeneous cone in the space \( \mathfrak{H} \) of hermitian matrices. Then the algebra of connectedness of the cone \( V(\mathfrak{A}) \) at the point \( e \in V(\mathfrak{A}) \) is a formally real Jordan algebra.

**Proof.** Let \( R(a, b, c) = g(e)(ab + ba, c) \). Then \( R(a, b, c) \) is symmetric because

\[
R(a, b, c) = R(b, a, c)
\]
and
\[ R(a, c, b) = g(e)(ac + ca, b) = tr((ac)b + (ca)b) = tr((ab + ba)c) \quad \text{(by axiom (4))} \]
\[ = g(e)(ab + ba, c) = R(a, b, c). \]

Moreover,
\[ tr(a \triangle(a \triangle a)) = tr(a(a \triangle a)) = tr(a(a^\wedge a + a_\vee a)) \]
\[ = tr(a^2(a^\wedge + a_\vee)) = tr((a^2)a) \]
\[ = g(e)(a^2, a) = \frac{1}{2}R(a, a, a) \]

Thus we get \( R(a, a, a) = -Q(a, a, a) \) and for all \( a, b, c \in \mathfrak{H}, \) we have
\[ g(e)(ab + ba, c) = R(a, b, c) = -Q(a, b, c) = 2g(e)(a \sqcap b, c) \]
by Lemma 3.1. Hence
\[ a \sqcap b = \frac{1}{2}(ab + ba). \]

Since the algebra \((\mathfrak{H}, \sqcap)\) is commutative, it suffices to show that \((\mathfrak{H}, \sqcap)\) satisfies the Jordan identity and formally real. If \( a, b, c \in \mathfrak{H}, \)
\[ tr((a^2 \sqcap(b \sqcap a))c) \]
\[ = \frac{1}{4}\{tr(a^2(ba)) + tr(a^2(ab)) + tr((ba)a^2) + tr((ab)a^2)\}c \]
\[ = \frac{1}{4}\{tr((a^2b)a) + tr((ba^2)a) + tr(a(a^2b)) + tr(a(ba^2))\}c \]
\[ = tr((a^2 \sqcap b) \sqcap a)c. \]

by axiom (3) and (4). Thus if \( a, b \in \mathfrak{H}, \) we have
\[ g(e)(a^2 \sqcap(b \sqcap a), c) = g(e)((a^2 \sqcap b) \sqcap a, c) \quad \text{for all } c \in \mathfrak{H}, \]
and so \( a^2 \sqcap(b \sqcap a) = (a^2 \sqcap b) \sqcap a. \)

Suppose that \( a \sqcap a + b \sqcap b = 0. \) Since \( a \sqcap a + b \sqcap b = a^2 + b^2 = 0, \)
\[ g(e)(a, a) + g(e)(b, b) = tr(a^2) + tr(b^2) = tr(a^2 + b^2) \]
\[ = g(e)(a^2 + b^2, e) = 0. \]

By axiom (5), if \( a \neq 0, \) then \( tr(a^2) > 0, \) and hence \( a = b = 0. \)
This completes the proofs. \( \square \)
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References


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