NEAR DUNFORD–PETTIS OPERATORS AND NRNP

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1. Preliminaries

Throughout this paper $X$ is a Banach space and $\mu$ is the Lebesgue measure on $[0,1]$ and all operators are assumed to be bounded and linear. $L^1(\mu)$ is the Banach space of all (classes of) Lebesgue integrable functions on $[0,1]$ with its usual norm. Let $T : L^1(\mu) \to X$ be an operator. Then

(a) $T$ is called representable if there exist $g : [0,1] \to X, ||g||_{\infty} < \infty$ such that $Tf = \int fg d\mu$ for all $f \in L^1(\mu)$.

(b) $T$ is a Dunford-Pettis operator if $T$ maps weakly compact sets into norm compact sets.

(c) $T$ is nearly representable if $T \cdot D : L^1(\mu) \to X$ is Bochner representable for every Dunford-Pettis operator $D : L^1(\mu) \to L^1(\mu)$. It is well known [1] that each bounded linear operator $T : L^1(\mu) \to X$ can be associated with a martingale $(\xi_n)$. The correspondence is

$$T(\psi) = \lim_{n \to \infty} \int \xi_n(t) \psi(t) \, dt$$

and

$$\xi_n = \sum_{E \in \Pi_n} \frac{T(\chi_E)}{\mu(E)} \chi_E$$

where $\Pi_n$ is nth dyadic partition of $[0,1]$, i.e.,

$$\Pi_n = \left\{ I_{n,k} \mid I_{n,k} = \left[ \frac{k-1}{2^n}, \frac{k}{2^n} \right), k = 1, 2, 3, \ldots, 2^n - 1 \right\} \cup I_{n,2^n},$$

$n = 0, 1, 2, \ldots$, and $I_{n,2^n} = \left[ \frac{2^n - 1}{2^n}, 1 \right]$. Also Bourgain showed the following fact on Dunford-Pettis operators and associated martingale.

Received January 10, 1994.
1991 AMS Subject Classification: 46.
Key words: Near Dunford-Pettis operators, near Radon-Nikodym property.
FACT 1.1. (a) A uniformly bounded $X$-valued martingale is Pettis-Cauchy iff the corresponding operator is Dunford-Pettis.

(b) The martingale $(\xi_n)$ is Pettis-Cauchy iff $\lim_{n \to \infty} \| \int \xi_n \psi_n \| = 0$, whenever $(\psi_n)$ is an $L^\infty$-bounded weakly null sequence in $L^1$.

Notations and symbols are standard and not appeared here can be seen in [2] and [3].

2. Near Dunford-Pettis operators and NRNP

The following fact was proved by Petrakis [5, P.27].

FACT 2.1. Let $T : L^1(\mu) \to X$ be any Dunford-Pettis operator. Then there is a non representable operator $S : L^1(\mu) \to L^1(\mu)$ such that $T \cdot S$ is a representable operator.

Since every representable operator is Dunford-Pettis operator, it is natural to ask if there exist non Dunford-Pettis operator which satisfies the Fact 2.1. The next theorem is a partial answer to this question.

THEOREM 2.2. Let $T : L^1(\mu) \to X$ be nearly representable. Then there exists a non Dunford-Pettis operator $S : L^1(\mu) \to L^1(\mu)$ such that $T \cdot S : L^1(\mu) \to X$ is representable.

Proof. Let $B_{L^1}$ be the unit ball of $L^1(\mu)$. Then we may assume that \( \frac{3}{2} \cdot T(B_{L^1}) \subseteq W \), where $W$ is an open ball of $X$. Since every nearly representable operator is Dunford-Pettis, as Petrakis showed, there exist a tree $(\psi_{n,k}), 1 \leq k \leq 2^n, n = 0, 1, 2, \ldots$ of functions in $L_{\infty}[0,1]$ and a system $(B_{n,k}), 1 \leq k \leq 2^n, n = 0, 1, 2, \ldots$ of open balls of $X$ such that

1. \( 1 \leq \| \psi_{n,k} \|_1 \leq 2 \), for \( 1 \leq k \leq 2^n, n = 0, 1, 2, \ldots \)
2. \( \| \psi_{n+1,2k-1} - \psi_{n+1,2k} \|_1 \geq \frac{1}{2} \) for \( 1 \leq k \leq 2^n, n = 0, 1, 2, \ldots \)
3. $B_{n,k}$ has center at $T(\psi_{n,k})$ and radius $r_{n,k}$ at most $2^{-n}$
4. $B_{n,k} \subseteq W$ for all $n, k$ and $B_{n+1,2k-1} \cup B_{n+1,2k} \subseteq B_{n,k}$

To construct the above tree and system, $\rho_n : [0,1] \to \{-1,1\}$ was defined as $\rho_n(\omega) = 1$ if $\omega \in \left[ \frac{2k}{2^n}, \frac{(2k+1)}{2^n} \right)$ and $\rho_n(\omega) = -1$ if $\omega \in \left( \frac{(2k+1)}{2^n}, \frac{(2k+2)}{2^n} \right)$
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\[ \left( \frac{2(k + 1)}{2^n}, \frac{2(k + 1)}{2^n} \right), \text{ for } k = 0, 1, 2, \ldots, 2^{n-1} - 1, \text{ or } \omega = 1. \] Then this \((\rho_n)\) is \(L^\infty\)-bounded weakly null sequence in \(L^1(\mu)\). And \(\psi_{n,k}\) was defined as \(\psi_{0,1} = \frac{3}{2} \chi_{[0,1]}, \psi_{n+1,2k-1} = \psi_{n,k}(1 + \frac{1}{3} \eta_k)\) and \(\psi_{n+1,2k} = \psi_{n,k}(1 - \frac{1}{3} \eta_k)\), where \((\eta_k)\) is a sequence of suitable functions in the sequence \((\rho_n)\). Then

\[ \left\| \sum_{k=1}^{2^n} (-1)^{k+1} \psi_{n,k} \right\| \geq 2^{n-1}, \]

for \(n = 1, 2, 3, \ldots\) Put \(\xi_n(t)(s) = 2^{-n} \sum_{k=1}^{2^n} h_{n,k}(t) \psi_{n,k}(s)\), where \(h_{n,k} = 2^n \chi_{I_{n,k}}\), then \((\xi_n)\) is a \(L^1\)-valued martingale associated to the \(\frac{1}{3}\)-tree \((\psi_{n,k})\). And \((\xi_n)\) is not convergent. Moreover we will show that \((\xi_n)\) is not Pettis-Cauchy. For this calculate \(\| \int \xi_n \rho_n \| \), then

\[
\left\| \int \xi_n \rho_n \right\| = \left\| \int \sum_{k=1}^{2^n} \chi_{I_{n,k}} \psi_{n,k} \rho_n(t) dt \right\| d(\mu) \\
= \left\| \sum_{k=1}^{2^n} \psi_{n,k} \int_{I_{n,k}} \rho_n(t) dt \right\| d(\mu) = 2^{-n} \left\| \sum_{k=1}^{2^n} (-1)^{k+1} \psi_{n,k} \right\| d(\mu) \geq \frac{1}{2}.
\]

This means that \(\lim_{n \to \infty} \| \int \xi_n \rho_n \| \neq 0\). Hence by the Fact 1.1 (b), \((\xi_n)\) is not Pettis-Cauchy. Again by the Fact 1.1 (a), the operator \(S : L^1(\mu) \to L^1(\mu)\) which is associated to this martingale \((\xi_n)\) is not a Dunford-Pettis operator. But the martingale \(T(\xi_n)\), which is associated with \(T \cdot S\), converges as we can see in [5]. Thus \(T \cdot S\) is representable and completes the proof.

Whenever a Banach space \(X\) and a nearly representable operator \(N : L^1(\mu) \to X\) are given, the Theorem 2.2 enables us to think of a new set of operators in \(B(L^1(\mu))\), the set of all bounded linear operators from \(L^1(\mu)\) into \(L^1(\mu)\). We will call such operators near Dunford-Pettis operators with respect to \(X\) and \(N\), and will denote them as \(NDP(N,X)\) operators. i.e.,

\[ NDP(N,X) = \{ T \in B(L^1(\mu)) \mid N \cdot T : L^1(\mu) \to X \text{is representable} \} \]

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REMARK 2.3. For every Banach space $X$ and nearly representable operator $N : L^1(\mu) \to X$, $NDP(N, X)$ contains all Dunford-Pettis operators in $B(L^1(\mu))$. Moreover as we can see in Theorem 2.2, the set $NDP(N, X)$ is strictly larger than the set of all Dunford-Pettis operators in $B(L^1(\mu))$. And as shown in [4] the Volterra operator $V : L^1(\mu) \to C[0, 1]$ is not representable. Hence the identity operator $I$ in $B(L^1(\mu))$ is not a $NDP(V, C[0, 1])$. Recently Kaufman, Petrakis, Riddle and Uhl introduced a new concept to characterize a Banach space which is called near Radon-Nikodym property space[4].

DEFINITION 2.4. A Banach space $X$ is said to have the near Radon-Nikodym property (NRNP) iff every nearly representable operator $N : L^1(\mu) \to X$ is representable.

The following theorem states a relation between $NDP(N, X)$ and NRNP of $X$.

THEOREM 2.5. Let $X$ be a Banach space. Then $X$ has the NRNP iff $B(L^1(\mu)) = NDP(N, X)$ for every nearly representable operator $N : L^1(\mu) \to X$.

Proof. Let $T \in B(L^1(\mu))$ and $N : L^1(\mu) \to X$ be any nearly representable operator. Then $N \cdot T : L^1(\mu) \to X$ is also nearly representable [5]. Hence $N \cdot T$ is representable. i.e., $T \in NDP(N, X)$. For the converse, let $N : L^1(\mu) \to X$ be a nearly representable operator. Then since $I : L^1(\mu) \to L^1(\mu)$ is $NDP(N, X)$ operator, $N = N \cdot I$ is representable. Thus $X$ has NRNP.

References

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