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NEAR DUNFORD–PETTIS OPERATORS AND NRNP

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1. Preliminaries

Throughout this paper X is a Banach space and μ is the Lebesgue measure on $[0,1]$ and all operators are assumed to be bounded and linear. $L^1(\mu)$ is the Banach space of all (classes of) Lebesgue integrable functions on $[0,1]$ with its usual norm. Let $T : L^1(\mu) \to X$ be an operator. Then

(a)T is called representable if there exist $g : [0,1] \to X, \|g\|_\infty < \infty$ such that $Tf = \int fgd\mu$ for all $f \in L^1(\mu)$.

(b) T is a Dunford-Pettis operator if T maps weakly compact sets into norm compact sets.

(c) T is nearly representable if $T \cdot D : L^1(\mu) \to X$ is Bochner representable for every Dunford-Pettis operator $D : L^1(\mu) \to L^1(\mu)$. It is well known [1] that each bounded linear operator $T : L^1(\mu) \to X$ can be asssociated with a martingale (ξ_n). The correspondence is

$$T(\psi) = lim_{n\to\infty} \int \xi_n(t)\psi(t)dt$$

and

$$\xi_n = \Sigma_{E\in\Pi_n} \frac{T(\chi_E)}{\mu(E)}\chi_E$$

where Π_n is nth dyadic partion of $[0,1]$, i.e.,

$$\Pi_n = \left\{I_{n,k} \mid I_{n,k} = \left[\frac{k-1}{2^n}, \frac{k}{2^n}\right), k = 1,2,3,\cdots,2^n-1\right\} \cup I_{n,2^n},$$

$n = 0,1,2,...$, and $I_{n,2^n} = \left[\frac{2^n-1}{2^n}, 1\right]$. Also Bourgain showed the following fact on Dunford-Pettis operators and associated martingale.

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FACT 1.1. (a) *A uniformly bounded X-valued martingale is Pettis-Cauchy iff the corresponding operator is Dunford-Pettis.*

(b) *The martingale* (ξ_n) *is Pettis-Cauchy iff* $lim_{n\to\infty}\|\int \xi_n\psi_n\| = 0$, *whenever* (ψ_n) *is an* L^∞*-bounded weakly null sequence in* L^1.

Notations and symbols are standard and not appeared here can be seen in [2] and [3].

2. Near Dunford-Pettis operators and NRNP

The following fact was proved by Petrakis [5, P.27].

FACT 2.1. *Let* $T : L^1(\mu) \to X$ *be any Dunford-Pettis operator. Then there is a non representable operator* $S : L^1(\mu) \to L^1(\mu)$ *such that* $T \cdot S$ *is a representable operator.*

Since every representable operator is Dunford-Pettis operator, it is natural to ask if there exist non Dunford-Pettis operator which satisfies the Fact 2.1. The next theorem is a partial answer to this question.

THEOREM 2.2. *Let* $T : L^1(\mu) \to X$ *be nearly representable. Then there exists a non Dunford-Pettis operator* $S : L^1(\mu) \to L^1(\mu)$ *such that* $T \cdot S : L^1(\mu) \to X$ *is representable.*

Proof. Let B_{L^1} be the unit ball of $L^1(\mu)$. Then we may assume that $\frac{3}{2} \cdot T(B_{L^1}) \subseteq W$, where W is an open ball of X. Since every nearly representable operator is Dunford-Pettis, as Petrakis showed, there exist a tree $(\psi_{n,k}), 1 \le k \le 2^n, n = 0,1,2,....$ of functions in $L_\infty[0,1]$ and a system $(B_{n,k}), 1 \le k \le 2^n, n = 0,1,2,....$ of open balls of X such that

(1) $1 \le\| \psi_{n,k} \|_1\le 2$, for $1 \le k \le 2^n, n = 0,1,2,\cdots$

(2) $\| \psi_{n+1,2k-1} - \psi_{n+1,2k} \|_1\ge \frac{1}{2}$ for $1 \le k \le 2^n, n = 0,1,2,\cdots$

(3) $B_{n,k}$ has center at $T(\psi_{n,k})$ and radius $r_{n,k}$ at most 2^{-n}

(4) $B_{n,k} \subseteq W$ for all n,k and $\bar{B}_{n+1,2k-1} \cup \bar{B}_{n+1,2k} \subseteq B_{n,k}$

To construct the above tree and system, $\rho_n : [0,1] \to \{-1,1\}$ was defined as $\rho_n(\omega) = 1$ if $\omega \in \left[\frac{2k}{2^n}, \frac{(2k+1)}{2^n}\right)$ and $\rho_n(\omega) = -1$ if $\omega \in$

$\left[\frac{(2k+1)}{2^n}, \frac{2(k+1)}{2^n}\right)$, for $k = 0, 1, 2, \cdots, 2^{n-1} - 1$, or $\omega = 1$. Then this (ρ_n) is L^∞ -bounded weakly null sequence in $L^1(\mu)$. And $\psi_{n,k}$ was defined as $\psi_{0,1} = \frac{3}{2}\chi_{[0,1]}, \psi_{n+1,2k-1} = \psi_{n,k}(1 + \frac{1}{3}\eta_k)$ and$\psi_{n+1,2k} = \psi_{n,k}(1 - \frac{1}{3}\eta_k)$, where (η_k) is a sequence of suitable functions in the sequence (ρ_n). Then

$$\left\|\sum_{k=1}^{2^n}(-1)^{k+1}\psi_{n,k}\right\| \geq 2^{n-1},$$

for $n = 1, 2, 3, \cdots$ Put $\xi_n(t)(s) = 2^{-n}\sum_{k=1}^{2^n} h_{n,k}(t)\psi_{n,k}(s)$, where $h_{n,k} = 2^n\chi_{I_{n,k}}$, then (ξ_n) is a L^1-valued martingale associated to the $\frac{1}{3}$−tree $(\psi_{n,k})$. And (ξ_n) is not convergent. Moreover we will show that (ξ_n) is not Pettis-Cauchy. For this calculate $\|\int \xi_n\rho_n\||$, then

$$\left\|\int \xi_n\rho_n\right\| = \int\left|\int\sum_{k=1}^{2^n}\chi_{I_{n,k}}\psi_{n,k}\rho_n(t)dt\right|d(\mu)$$

$$= \int\left|\sum_{k=1}^{2^n}\psi_{n,k}\int_{I_{n,k}}\rho_n(t)dt\right|d(\mu) = 2^{-n}\int\left|\sum_{k=1}^{2^n}(-1)^{k+1}\psi_{n,k}\right|d(\mu) \geq \frac{1}{2}.$$

This means that $lim_{n\to\infty}\|\int\xi_n\rho_n\| \neq 0$. Hence by the Fact 1.1 (b), (ξ_n) is not Pettis-Cauchy. Again by the Fact 1.1 (a), the operator $S : L^1(\mu) \to L^1(\mu)$ which is associated to this martingale (ξ_n) is not a Dunford-Pettis operator. But the martingale $T(\xi_n)$, which is associated with $T \cdot S$, converges as we can see in [5]. Thus $T \cdot S$ is representable and completes the proof.

Whenever a Banach space X and a nearly representable operator $N : L^1(\mu) \to X$ are given, the Theorem 2.2 enables us to think of a new set of operators in $B(L^1(\mu))$, the set of all bounded linear operators from $L^1(\mu)$ into $L^1(\mu)$. We will call such operators near Dunford-Pettis operators with respect to X and N, and will denote them as $NDP(N, X)$ operators. i.e.,

$$NDP(N, X) = \{T \in B(L^1(\mu)) \mid N \cdot T : L^1(\mu) \to X is representable\}$$

REMARK 2.3. For every Banach space X and nearly representable operator $N : L^1(\mu) \to X$, $NDP(N,X)$ contains all Dunford-Pettis operators in $B(L^1(\mu))$. Moreover as we can see in Theorem 2.2, the set $NDP(N,X)$ is strictly larger than the set of all Dunford-Pettis operators in $B(L^1(\mu))$. And as shown in [4] the Volterra operator $V : L^1(\mu) \to C[0,1]$ is not representable. Hence the identity operator I in $B(L^1(\mu))$ is not a $NDP(V, C[0,1])$. Recently Kaufman, Petrakis, Riddle and Uhl introduced a new concept to characterize a Banach space which is called near Radon-Nikodym property space[4].

DEFINITION 2.4. A Banach space X is said to have the near Radon-Nikodym property (NRNP) iff every nearly representable operator $N : L^1(\mu) \to X$ is representable.

The following theorem states a relation between $NDP(N,X)$ and NRNP of X.

THEOREM 2.5. *Let X be a Banach space. Then X has the NRNP iff $B(L^1(\mu)) = NDP(N,X)$ for every nearly representable operator$N : L^1(\mu) \to X$.*

Proof. Let $T \in B(L^1(\mu))$ and $N : L^1(\mu) \to X$ be any nearly representable operator. Then $N \cdot T : L^1(\mu) \to X$ is also nearly representable [5]. Hence $N \cdot T$ is representable. i.e., $T \in NDP(N,X)$. For the converse, let $N : L^1(\mu) \to X$ be a nearly representable operator. Then since $I : L^1(\mu) \to L^1(\mu)$ is $NDP(N,X)$ operator, $N = N \cdot I$ is representable. Thus X has NRNP.

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