A NONEXISTENCE THEOREM FOR STABLE EXPONENTIALLY HARMONIC MAPS

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1. Introduction

Let $M$ and $N$ be compact Riemannian manifolds and $f : M \to N$ be a smooth map. Following J. Eells, $f$ is exponentially harmonic if it represents a critical point of the exponential energy integral

$$ E(f) = \int_M \exp(\|df\|^2) \, dM $$

where $\|df\|^2$ is the energy density defined as $\sum_{i=1}^m \|df(e_i)\|^2$, $m = \dim M$, for orthonormal frame $e_i$ of $M$. The Euler-Lagrange equation of the exponential energy functional $E$ can be written

$$ \exp(\|df\|^2)(\tau(f) + df(\nabla\|df\|^2)) = 0 $$

where $\tau(f)$ is the tension field along $f$. Hence, if the energy density is constant, every harmonic map is exponentially harmonic and vice versa.

An exponentially harmonic map is called stable if it represents, furthermore, a local minimum point of the exponential energy. When the target manifold is the standard sphere $S^n$, it is well known that every stable harmonic map $f : M \to S^n$, $n \geq 3$, is constant [2]. This is not the case with exponentially harmonic maps since every identity map of $M$ is a stable exponentially harmonic map [1]. In this note, however, we can prove the following

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THEOREM. Let $M$ be a compact Riemannian manifold of dimension $m$. Every nonconstant exponentially harmonic map $f : M \to S^n$ is unstable if $\|df\|^2(x) < n - 2$ for every $x \in M$.

2. Proof

We begin with a property of conformal vector fields on $S^n$. Embed $S^n$ canonically into $\mathbb{R}^{n+1}$. Let $\phi : \mathbb{R}^{n+1} \to \mathbb{R}$ be a linear map with $\phi(O) = O$, $\|\nabla \phi\| = 1$. Set $V_\phi = \nabla \phi$ and let $V^t_\phi$ and $V^n_\phi$ be tangential and normal components of $V_\phi$, respectively and $\nu$ be the outward unit normal vector to $S^n$, then we have the following lemma. For proof, refer, for example, to [2].

**Lemma.** For any vector $X$ tangent to $S^n$,

$$\nabla_X V^t_\phi = -(V_\phi, \nu) X.$$

Let $f : M \to S^n$ be a nonconstant exponentially harmonic map. Consider $V^t_\phi$ and the variation of $f$ in the direction of $V^t_\phi$, then by the second variation formula for $E$ (cf. [1]) we have

$$\frac{d^2 E_\phi}{dt^2}(0) = \int_M \exp(\|df\|^2) \sum_{i=1}^m \langle \nabla df(e_i) V^t_\phi, df(e_i) \rangle^2$$

$$+ \|\nabla df(e_i) V^t_\phi\|^2 - \langle R(df(e_i), V^t_\phi) df(e_i), V^t_\phi \rangle \, dM$$

where $R$ is the curvature tensor of $S^n$. Write $\frac{d^2 E_\phi}{dt^2}(0)$ as

$$\frac{d^2 E_\phi}{dt^2}(0) = \int_M \exp(\|df\|^2) \left( \sum_{i=1}^m ((A) + (B) - (C)) \right) \, dM$$

and now calculate. By Lemma,

$$\|\nabla df(e_i) V^t_\phi\|^2 = (V_\phi, \nu)^2 \|df(e_i)\|^2$$

and since

$$\langle R(df(e_i), V^t_\phi) df(e_i), V^t_\phi \rangle = \|df(e_i)\|^2 \|V^t_\phi\|^2 - \langle df(e_i), V^t_\phi \rangle^2$$
A nonexistence theorem for stable exponentially harmonic maps

we have
\[ \sum_{i=1}^{m} ((B) - (C)) \]
\[ = \sum_{i=1}^{m} (\|df(e_i)\|^2 \langle (V_{\phi_i}^t, \nu)^2 - \|V_{\phi_i}^t\|^2 \rangle + \langle df(e_i), V_{\phi_i}^t \rangle^2) \]
\[ := \sum_{i=1}^{m} (D). \]

Consider now \( n + 1 \) linear functions \( \phi_j \) such that \( V_{\phi_j}^t := V_j \) form an orthonormal basis of \( \mathbb{R}^{n+1} \) and calculate \( \sum_{j=1}^{n+1} \frac{d^2 E_{\phi_j}}{dt^2} (0) \). From
\[ \sum_j \langle df(e_i), V_j^t \rangle^2 = \sum_j \langle df(e_i), V_j \rangle^2 = \| df(e_i) \|^2, \]
\[ \sum_j (V_j, \nu)^2 = \| \nu \|^2 = 1, \]
\[ \sum_j ((V_j, \nu)^2 - \| V_j^t \|^2) = \sum_j (\| V_j^n \|^2 - \| V_j \|^2) \]
\[ = \sum_j (2 \| V_j^n \|^2 - \| V_j \|^2) = 2 - (n + 1) \]
we have
\[ \sum_{j=1}^{n+1} \int_M \exp(\| df \|^2) \sum_{i=1}^{m} (D) dM = (2 - n) \int_M \| df \|^2 \exp(\| df \|^2) dM, \]
and if \( \| df \|^2 < n - 2 \), we have
\[ \sum_{j=1}^{n+1} \sum_{i=1}^{m} \langle \nabla df(e_i), V_{\phi_j}^t, df(e_i) \rangle^2 = \sum_j \sum_i (V_{\phi_j}, \nu)^2 \| df(e_i) \|^4 \]
\[ = \sum_i \| df(e_i) \|^4 \sum_j (V_{\phi_j}, \nu)^2 = \sum_i \| df(e_i) \|^4 \]
\[ \leq \left( \sum_i \| df(e_i) \|^2 \right)^2 = \| df \|^4 < (n - 2) \| df \|^2 \]
and consequently we have

$$\sum_{j=1}^{n+1} \int_M \exp(\|df\|^2) \sum_{i=1}^{m} (A_i) \, dM < (n-2) \int_M \|df\|^2 \exp(\|df\|^2) \, dM.$$ 

Hence, we have finally,

$$\sum_{j=1}^{n+1} \frac{d^2 E_{\phi_j}}{dt^2}(0) = \sum_{j=1}^{n+1} \int_M \exp(\|df\|^2) \left( \sum_{i=1}^{m} ((A_i) + (B_i) - (C_i)) \right) \, dM < 0.$$ 

Therefore, at least one $\frac{d^2 E_{\phi_j}}{dt^2}(0)$ should be negative, that is, a non-constant exponentially harmonic map $f$ with $\|df\|^2 < n-2$ is not stable. This completes the proof.

References


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