Bull. Korean Math. Soc. **32** (1995), No. 2, pp. 221–231

A UNIFORM LAW OF LARGE NUMBERS FOR PRODUCT RANDOM MEASURES

BYUNG MUN KIL AND JOONG SUNG KWON

1. Introduction

Let $Z_1, Z_2, \cdots, Z_l$ be random set functions or integrals. Then it is possible to discuss their products. In the case of random integrals, Z_i is a random set function indexed by a family, $\mathcal{G}_i$ say, of real valued functions g on S_i for which the integrals $Z_i(g) = \int g dZ_i$ are well defined. If $g_i \in \mathcal{G}_i$ $(i = 1, 2 \ldots l)$ and $g = g_1 \otimes \cdots \otimes g_l$ denotes the tensor product $g(s) = g_1(s_1)g_2(s_2)\cdots g_l(s_l)$ for $s = (s_1, s_2, \cdots, s_l)$ and $s_i \in S_i$, then we can define $Z(g) = (Z_1 \times Z_2 \times \cdots \times Z_l)(g) = Z_1(g_1)Z_2(g_2)\cdots Z_l(g_l)$. Write $\mathcal{G}_1 \otimes \mathcal{G}_2 \otimes \cdots \otimes \mathcal{G}_l = \{g_1 \otimes g_2 \otimes \cdots \otimes g_l : g_i \in \mathcal{G}_i, i = 1, 2 \cdots l\}$ for the set of tensor products. The questions we are interested in are the followings: Characterize those index families $\mathcal{G}$ on the product space $S = S_1 \times S_2 \times \cdots \times S_l$ that contain $\mathcal{G}_1 \otimes \mathcal{G}_2 \otimes \cdots \otimes \mathcal{G}_l$, for which a *regular* extension of $Z = Z_1 \times Z_2 \times \cdots \times Z_l$ exists. Here *regular* can mean many things, for examples, boundedness or some continuity of sample paths, or LLN or weak convergence for some sequences of those products. In this paper we will restrict ourselves to the case when $S_i = \mathbf{I}^{d_i}$, d_i−dimensional unit cube and $\mathcal{G}_i = \mathcal{B}(\mathbf{I}^{d_i})$, Borel σ−field on $\mathbf{I}^{d_i}$. In this case Z_i and Z are called random measure and product random measure respectively. Under the setups in this paper we will state and prove uniform law of large numbers for some product random measures under some conditions on index families. The products considered here are those of empirical, Poisson and partial sum processes. And the condition used to restrict index families is the *smooth boundary*

Received January 26, 1994.
1991 AMS Subject Classification: 60F05, 60G05.
Key words: Law of large numbers, Product partial sum process, Product empirical process, Product Poisson process.

condition, which was invented and used to prove the same question for set indexed partial sum processes in Bass and Pyke(1984).

Let $\mathcal{A}$ be a sub-family of $\mathcal{B}(\mathbf{I}^d)$ with $d = \sum_{j=1}^{l} d_j$. Given $A \subset \mathbf{I}^d$, let $A(\delta) = \{x : \rho(x, \partial A) < \delta\}$ be the δ-annulus of ∂A, where $\rho(\cdot,\cdot)$ is the Euclidean distance and ∂ denotes the Euclidean boundary of A. $\mathcal{A}$ is said to satisfy **Assumption SBC**(Smooth Boundary Condition) if

$$r(\delta) := \sup_{A \in \mathcal{A}} |A(\delta)| \to 0 \text{ } as \text{ } \delta \to 0.$$

If, for example, $\mathcal{A}$ were the collection of convex subsets of $\mathbf{I}^d$, it is known to satisfy **SBC**.

We prove the following:

THEOREM. *Let $Z_1, Z_2, \cdots, Z_l$ be random measures on $\mathcal{B}(\mathbf{I}^{d_i})$ with $i = 1, 2, \cdots, l$. Let $\mathcal{A}$ be a subfamily of $\mathcal{B}(\mathbf{I}^d)$ satisfying SBC with $d = \sum_{j=1}^{j=l} d_j$. Let F be a (deterministic) set function defined on $\mathcal{A}$. And finally let $\{f_n\}$ denote a sequence of some real valued (measurable) functions defined on $\mathbf{R}^l$. Then, with probability 1*

$$\|f_n(Z_1 \times Z_2 \times \cdots \times Z_l) - F\|_{\mathcal{A}} \to 0$$

as $n \to \infty$.

2. Product Empirical Measures

Let $\{X_{ij} : i \in N\}_{j=1}^{j=l}$ be families of random variables with row-wise common distributions G_j, on $\mathbf{I}^{d_j}$ respectively. Note that we are not assuming any relations among sequences. Then the product empirical measure corresponding to $\{X_{ij}\}$, indexed by subsets of $\mathbf{I}^d$ with $d = \sum_{j=1}^{j=l} d_j$, is defined by

$$\begin{aligned} F_n(A) :=& n^{-d} \sharp\{(k_1, k_2, \cdots, k_l) : (X_{k_1}, X_{k_2}, \cdots, X_{k_l}) \\ & \in A, k_j \le n, j = 1, \cdots, l\} \\ =& \sum_{k_j \le n, j=1,2,\cdots,l} n^{-d} \delta_{(X_{k_1}, X_{k_2}, \cdots, X_{k_l})}(A) \end{aligned}$$

where $\delta_{(x_1,x_2,\cdots,x_l)}$ is the degenerate probability measure that gives measure 1 to the point $(x_1, x_2, \cdots, x_l)$. We view here and hereafter F_n is defined on $\mathcal{B}(\mathbf{I}^d)$ even if it is well defined on any subset A of $\mathbf{I}^d$.

To prove the main theorems in this section we need the following straightforward consequence of known results. We denote F the Lebesgue measure on $\mathcal{B}(\mathbf{I}^d)$.

LEMMA 2.1. *Let A be rectilinear form of subsets of $\mathbf{I}^d$ ($n^d F_n(A)$ is a $B(n^d, F(A))$ Bernoulli r.v. with $p = F(A)$). Then, with probability 1*

$$F_n(A) \to F(A)$$

as $n \to \infty$.

Proof. Let $\mathbf{x} = (x_1, x_2, \cdots, x_d)$ be fixed and write $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_l)$ where $\mathbf{x}_1 = (x_1, x_2, \cdots, x_{d_1})$, $\mathbf{x}_2 = (x_1, x_2, \cdots, x_{d_2})$ and $\mathbf{x}_l = (x_1, x_2, \cdots, x_{d_l})$. Then $(0, \mathbf{x}] = \{(y_1, y_2, \cdots, y_d) | 0 < y_i \le x_i, \ \ i = 1, 2, \cdots, d\}$ and $|(0, \mathbf{x}]| = |(0, \mathbf{x}_1]||(0, \mathbf{x}_2]| \cdots |(0, \mathbf{x}_l]|$. And by classical Glivenko-Cantelli theorem we have $F_n(0, \mathbf{x}] \to F(0, \mathbf{x}]$. Since any rectilinear form can be approximated by a finite number of unions and finite differences of rectangles of the form $(0, \mathbf{x}]$, we have by linearity, $F_n(A) \to F(A)$.

Before proving the theorem we introduce some notation following Bass and Pyke (1984), which will be used throughout the paper. Let m be a fixed positive integer and partition $\mathbf{I}^d$ into regular cubes of side length $1/m$. Let $C_\mathbf{j} = \frac{1}{m}(\mathbf{j} - \mathbf{1}, \mathbf{j}]$, where $\mathbf{j} = (j_1, j_2, \cdots, j_d)$ and $\mathbf{1} = (1, 1, \cdots, 1)$ with $1 \le j_k \le m$. Then for any $A \in \mathcal{A}$, define

$$R_m^-(A) = \cup_{C_\mathbf{j} \subset A} C_\mathbf{j}, \qquad \text{and} \qquad R_m^+(A) = \cup_{C_\mathbf{j} \cap A = \emptyset} C_\mathbf{j}.$$

That is, $R_m^-(A)$ and $R_m^+(A)$ are the inner and the outer rectilinear fits of A by cubes of side length $1/m$. Then since the furthest any point of $R_m^+(A) \setminus R_m^-(A)$ can be from the boundary of A is the diameter of a cube of size $1/m$,by the smooth boundary condition, we have

$$\sup_{A \in \mathcal{A}} |R_m^+(A) \setminus R_m^-(A)| \le r(d^{1/2}/m). \tag{2.1}$$

Now define

$$\mathcal{R}_m^- = \{R_m^-(A) | A \in \mathcal{A}\},$$

and

$$\mathcal{R}_m^\triangle = \{R_m^+(A) \setminus R_m^-(A) | A \in \mathcal{A}\}.$$

Then, since m is finite, $\sharp(\mathcal{R}_m^-)$ and $\sharp(\mathcal{R}_m^\triangle)$ are finite respectively.

Now we are ready to state and prove the strong law of large numbers for product empirical measures.

THEOREM 2.2. (Uniform Case) *If G_i are uniform on $\mathbf{I}^{d_i}$ with $i = 1, 2, \cdots, l$. And $\mathcal{A}$ satisfies SBC. Then with probability* 1,

$$\sup_{A \in \mathcal{A}} |F_n(A) - F(A)| \longrightarrow 0,$$

as $n \to \infty$

Proof.

$$\begin{aligned}
&\sup_{A \in \mathcal{A}} |F_n(A) - F(A)| \\
:=&\|F_n - F\|_{\mathcal{A}} \le \|F_n(R_m^-(\cdot)) - F(R_m^-(\cdot))\|_{\mathcal{A}} \\
&+ \|F_n(\cdot) - F_n(R_m^-(\cdot))\|_{\mathcal{A}} + \|F(\cdot) - F(R_m^-(\cdot))\|_{\mathcal{A}} \\
\le&\|F_n - F\|_{\mathcal{R}_m^-} + \|F_n\|_{\mathcal{R}_m^\triangle} + \|F\|_{\mathcal{R}_m^\triangle}.
\end{aligned}$$

As $n \to \infty$, Since $\sharp\mathcal{R}_m^- < \infty$ and $\sharp\mathcal{R}_m^\triangle < \infty$ the last two bounds converges a.s. to $2\|F\|_{\mathcal{R}_m^\triangle}$ by lemma 2.1 and the classical Glivenko-Cantelli theorem. Again by lemma 2.1 $\|F_n - F\|_{\mathcal{R}_m^-} \to 0$. Finally since $\sup_{A \in \mathcal{A}} |R_m^+(A) \setminus R_m^-(A)| \le r(\sqrt{d}/m) \to 0$ as $m \to \infty$, we have $\|F\|_{\mathcal{R}_m^\triangle} \to 0$ as $m \to \infty$.

We call $\mathcal{A}$ to satisfy totally bounded with inclusion if for all $\delta > 0$ there exists a finite $\delta-$net $\mathcal{A}_\delta \subset \mathcal{A}$ such that for each $A \in \mathcal{A}$ there exist A_δ and A_δ^+ in $\mathcal{A}_\delta$ such that $A_\delta \subset A \subset A_\delta^+$ and $d(A_\delta^+, A_\delta) < \delta$.

THEOREM 2.3. (General case) *If G_i are distributions (not necessarily uniform) on $\mathbf{I}^{d_i}$ with $i = 1, 2, \cdots, l$ and $\mathcal{A}$ satisfies totally bounded with inclusion with respect to the metric $d_F(d_F(A, B) = F(A \triangle B))$.*

And Assume that every member of δ-net $\mathcal{A}_\delta$ is of the rectilinear form. Then with probability 1

$$\sup_{A\in\mathcal{A}} |F_n(A) - F(A)| \to 0.$$

as $n \to \infty$.

Proof. From the rectilinear assumption, above lemma 2.1 will also be true in this case. Hence

$$\|F_n - F\|_{\mathcal{A}} \le \sup_{A\in\mathcal{A}} |F_n(A_\delta) - F(A_\delta)| + \sup_{A\in\mathcal{A}} |F_n(A_\delta^+ \setminus A_\delta)| + \sup_{A\in\mathcal{A}} |F(A_\delta^+ \setminus A_\delta)|$$

Since all sup are in fact on finite number of terms for each $\delta > 0$ and $F(A_\delta^+ \setminus A_\delta) = d_F(A_\delta^+, A_\delta)$, by above statement, we have $\|F_n - F\|_{\mathcal{A}} \to 0$ a.s. as $n \to \infty$.

3. Product Poisson Measures

Let Y_i be Poisson measures with integer parameters λ_i on $\mathcal{B}(\mathbf{I}^{d_i})$ with $i = 1, 2, \cdots, l$. Note that for notational convenience the parameters are not included in the Y's. Let $d = \sum_{j=1}^{j=l} d_j$ and let $\{U_{i_j} : i \in \mathbf{N}, j = 1, 2, \cdots, l\}$ (indicate the location of random points) denote sequences of independent uniformly distributed random variables on $\mathbf{I}^{d_i}$ respectively. The product Poisson measure of Y_i is defined as, for $B \in \mathcal{B}(\mathbf{I}^d)$,

$$Y_1 \times Y_2 \times \cdots \times Y_l(B) = \sum_{i_1=1}^{N_1} \sum_{i_2=1}^{N_2} \cdots \sum_{i_l=1}^{N_l} \delta_{(U_{i_1}, U_{i_2}, \cdots, U_{i_l})}(B),$$

where $N_i = Y_i(\mathbf{I}^{d_i})$ (indicates the number of random points) denote Poisson random variables with parameters λ_i.

Now we state and prove the strong law of large numbers for a sequence of products of Poisson measures under condition **SBC** on index family.

THEOREM 3.1. *Let Y_i be Poisson measures with integer parameters λ_i on $\mathcal{B}(\mathbf{I}^{d_i})$ with $i = 1, 2, \cdots, l$. Assume that $\mathcal{A}$ satisfy Assumption SBC. Then*

$$\text{(I)} \left\| \frac{Y_1 \times Y_2 \times \cdots \times Y_l(\cdot)}{\lambda_1 \lambda_2 \cdots \lambda_l} - |\cdot| \right\|_{\mathcal{A}} \longrightarrow 0, \ a.s. \quad \text{as} \quad \lambda_1, \lambda_2, \cdots, \lambda_l \longrightarrow \infty,$$

$$\text{(II)} \left\| \frac{Y_1 \times Y_2 \times \cdots \times Y_l(\cdot)}{N_1 N_2 \cdots N_l} - |\cdot| \right\|_{\mathcal{A}} \longrightarrow 0, \ a.s. \quad \text{as} \quad \lambda_1, \lambda_2, \cdots, \lambda_l \longrightarrow \infty,$$

where $N_i = Y_i(\mathbf{I}^{d_i})$ and $|\cdot|$ denotes the Lebesgue measure.

LEMMA 3.2. *Let A be rectilinear as defined above, then*

(i) $\frac{Y_i(A_{ix})}{\lambda_i} \longrightarrow |A_{ix}|$ a.s. as $\lambda_i \longrightarrow \infty$.

(ii) $\frac{Y_i(A_{ix})}{N_i} \longrightarrow |A_{ix}|$ a.s. as $\lambda_1 \longrightarrow \infty$.

(iii) $\frac{Y_1 \times Y_2 \times \cdots \times Y_l(A)}{\lambda_1 \lambda_2 \cdots \lambda_l} \longrightarrow |A|$ a.s. as $\lambda_1, \lambda_2, \cdots, \lambda_l \longrightarrow \infty$.

(iv) $\frac{Y_1 \times Y_\times \cdots \times Y_l(A)}{N_1 N_2 \cdots N_l} \longrightarrow |A|$ a.s. as $\lambda_1, \lambda_2, \cdots, \lambda_l \longrightarrow \infty$,

where $A_{ix} = \{y \in \mathbf{I}^{d_i} : (x, y) \in A\}$ is the d_i- dimensional section of A. And if $N_1 = 0$ or $N_2 = 0, \cdots$, or $N_l = 0$ then we define $(Y_1 \times Y_2 \times \cdots \times Y_l)/N_1 N_2 \cdots N_l = 1$ by convention.

Proof. The proof is straightforward. For (i), since the set structure is irrelevant, it suffices to show that $X(n)/n \to 1$ a.s. as $n \to \infty$ over the integers, where $X(n)$ is a Poisson random variable with parameter n. And this is a consequence of the Hsu-Robbins SLLN (Hsu and Robbins(1947)) since each $X(n)$ can be expressed as a sum of n independent Poisson random variables with parameter 1, which have a finite second moment.

For (ii),

$$\frac{Y_i(A_{ix})}{N_i} = \frac{Y_i(A_{ix})}{\lambda_i} \cdot \frac{\lambda_i}{N_i}.$$

Since $\lambda_i / N_i \to 1$ a.s., as $\lambda_i \to \infty$ (ii) follows from (i).

For (iii) and (iv), it suffices to prove it when A is a rectangle and this is done in (i) and (ii).

Proof of theorem 3.1. It suffices to consider when $l = 2$. First let

$m > 0$ be fixed. Since

$$
\begin{aligned}
&\limsup_{\lambda_1,\lambda_2\to\infty, A\in\mathcal{A}} \left| \frac{Y_1\times Y_2(A)}{\lambda_1\lambda_2} - |A| \right| \\
\le &\limsup_{\lambda_1,\lambda_2\to\infty, A\in\mathcal{A}} \left| \frac{Y_1\times Y_2(A)}{\lambda_1\lambda_2} - \frac{Y_1\times Y_2(R_m^-(A))}{\lambda_1\lambda_2} \right| \\
&+ \limsup_{\lambda_1,\lambda_2\to\infty, A\in\mathcal{A}} \left| \frac{Y_1\times Y_2(R_m^-(A))}{\lambda_1\lambda_2} - |R_m^-(A)| \right| \\
&+ \limsup_{\lambda_1,\lambda_2\to\infty, A\in\mathcal{A}} |A\setminus R_m^-(A)| \\
:= &T_1 + T_2 + T_3, \qquad (3.1)
\end{aligned}
$$

it remains to show that each of T_1, T_2 and $T_3 \to 0$ as $m\to\infty$. Consider T_1 and T_2

$$
\begin{aligned}
T_1 &\le \limsup_{\lambda_1,\lambda_2\to\infty, A\in\mathcal{A}} \left| \frac{Y_1\times Y_2(R_m^+(A)\setminus R_m^-(A))}{\lambda_1\lambda_2} \right| \\
&= \limsup_{\lambda_1,\lambda_2\to\infty, B\in\mathcal{R}_m^\triangle} \left| \frac{Y_1\times Y_2(R_m^+(A)\setminus R_m^-(A))}{\lambda_1\lambda_2} \right| \\
&= \max_{B\in\mathcal{R}_m^\triangle} |B| \\
&\le r(d^{1/2}/m) \qquad \text{a.s.},
\end{aligned}
$$

the second to last line following from lemma 3.2 (iii). Also by (iii) of lemma 3.2,

$$
T_2 \le \limsup_{\lambda_1,\lambda_2\to\infty} \max_{B\in\mathcal{R}_m^-} \left| \frac{Y_1\times Y_2(B)}{\lambda_1\lambda_2} - |B| \right| = 0 \qquad \text{a.s.}
$$

Finally notice that, by (2.1), $(T_3) \le r(d^{1/2}/m)$. Thus by (3.1),

$$
\limsup_{\lambda_1,\lambda_2\to\infty, A\in\mathcal{A}} \left| \frac{Y_1\times Y_2(A)}{\lambda_1\lambda_2} - |A| \right| \le 2r(d^{1/2}/m) \qquad \text{a.s.},
$$

which goes to zero as $m\to\infty$ by Assumption SBC.

For the proof of (II), use (iv) of lemma 3.2 and follow the proof of (I).

COROLLARY 3.3. *If* $\inf_{A\in\mathcal{A}} |A| > 0$, *then*

$$\text{(i)} \quad \limsup_{\lambda_1,\lambda_2,\cdots,\lambda_l\to\infty, A\in\mathcal{A}} \left| \frac{Y_1 \times Y_2 \times \cdots \times Y_l(A)}{\lambda_1\lambda_2\cdots\lambda_l|A|} - 1 \right| = 0 \qquad \text{a.s.}$$

$$\text{(ii)} \quad \limsup_{\lambda_1,\lambda_2\cdots,\lambda_l\to\infty, A\in\mathcal{A}} \left| \frac{Y_1 \times Y_2 \times \cdots \times Y_l(A)}{N_1N_2\cdots N_l|A|} - 1 \right| = 0 \quad \text{a.s.}.$$

Proof. This follows from theorem 3.1 by observing

$$\limsup_{\lambda_1,\lambda_2,\cdots,\lambda_l\to\infty, A\in\mathcal{A}} \left| \frac{Y_1 \times Y_2 \times \cdots \times Y_l(A)}{\lambda_1\lambda_2\cdots\lambda_l|A|} - 1 \right|$$

$$\leq \limsup_{A\in\mathcal{A}} |A|^{-1} \limsup_{\lambda_1,\lambda_2,\cdots,\lambda_l\to\infty, A\in\mathcal{A}} \left| \frac{Y_1 \times Y_2 \times \cdots \times Y_l(A)}{\lambda_1\lambda_2\cdots\lambda_l} - |A| \right| = 0 \text{ a.s.}$$

REMARK 3.4. In the above we have restricted the parameters to be discrete valued. However the result will also hold for continuous parameters if we impose some further structure. In particular let Y_{iR} be Poisson measures with parameter 1 defined on $[0,\infty)^{d_i}$, $d_i \geq 1$ and $i = 1, 2, \cdots, l$. Now suppose that the measures Y_i's in theorem 3.1 are defined, for $A_i \in \mathcal{B}(\mathbf{I}^{d_i})$ by

$$Yi(A_i) = Y_{iR}({\lambda_i}^{1/d_i} A_i),$$

then Y_i's are Poisson measures with the right parameters and in this case theorem 3.1 also can be shown to hold.

4. Product Partial Sum Processes

Let $\{X_{\mathbf{i}_j} : \mathbf{i}_j \in \mathrm{N}^{d_j}\}_{j=1}^{j=l}$ be families of sequence of random variables. Note that we are not assuming anything among sequences. Then the product partial sum process corresponding to $\{X_{\mathbf{i}_j} : \mathbf{i}_j \in \mathrm{N}^{d_j}\}_{j=1}^{j=l}$, indexed by subsets of $\mathbf{I}^d$ with $d = \sum_{j=1}^{j=l} d_j$, is defined by, for $A \subset \mathbf{I}^d$

$$S_n(A) := S_n(X_j, A)$$

$$:= \sum_{|\mathbf{i}_j|\leq n, j=1,2,\cdots,l} X_{\mathbf{i}_1}X_{\mathbf{i}_2}\cdots X_{\mathbf{i}_l}\delta_{(\mathbf{i}_1/n, \mathbf{i}_2/n,\cdots,\mathbf{i}_l/n)}(A),$$

where, $|\mathbf{i}_j|$ denotes the maximum of the components of $\mathbf{i}_j$ and $\delta_{(\mathbf{i}_1/n, \mathbf{i}_2/n, \cdots, \mathbf{i}_l/n)}(A) = 1$ or 0 depending on $(\mathbf{i}_1/n, \mathbf{i}_2/n, \cdots, \mathbf{i}_l/n) \in A$ or not. For partial sum processes, law of large number results have been shown to hold; see Bass and Pyke and Giné and Zinn.

In this section we prove a law of large numbers for a sequence of product partial sum processes $\{S_n(X_j, A) : A \in \mathcal{A}\}$under SBC condition on the index family $\mathcal{A}$.

THEOREM 4.1. *Let $\{X_{\mathbf{i}_j} : \mathbf{i}_j \in \mathbf{N}^{d_j}\}_{j=1}^{j=l}$ be families of sequences of random variables with $EX_j = \mu_j$, $E|X_j| < \infty$. Assume that for each j, $\{X_{\mathbf{i}_j} : \mathbf{i}_j \in \mathbf{N}^{d_j}\}$ satisfies strong law of large numbers. Then, under Assumption SBC on $\mathcal{A}$, we have*

$$\left\|n^{-d} S_n - \mu_1 \mu_2 \cdots \mu_l\right\|_{\mathcal{A}} \longrightarrow 0 \qquad a.s., \qquad as \qquad n \to \infty.$$

For the proof of theorem 4.1 we prove the following preliminary lemma. Recall the definition of R_m^-, R_m^+, $\mathcal{R}_m^-$ and $\mathcal{R}_m^{\triangle}$ from section 2.

LEMMA 4.2. *Let A be a rectilinear subset of $\mathbf{I}^d$. Then, with probability one, as $n \to \infty$,*

$$n^{-d} S_n(A) \longrightarrow \mu_1 \mu_2 \cdots \mu_l |A|.$$

Proof. Under the same development as in lemma 2.1 we have $|(0, \mathbf{x}]| = |(0, \mathbf{x}_1]||(0, \mathbf{x}_2]| \cdots |(0, \mathbf{x}_l]|$.

Now

$$\frac{S_n((0, \mathbf{x}])}{n^d} = \frac{\sharp(\mathbf{N}^d \cap n(0, \mathbf{x}])}{n^d} \cdot \frac{S_n((0, \mathbf{x}])}{\sharp(\mathbf{N}^d \cap n(0, \mathbf{x}])}.$$

Since

$$\begin{aligned} S_n((0, \mathbf{x}]) &= S_n((0, \mathbf{x}_1] \times (0, \mathbf{x}_2] \times \cdots (0, \mathbf{x}_l]) \\ &= S_{1n}((0, \mathbf{x}_1]) S_{2n}((0, \mathbf{x}_2]) \cdots S_{ln}((0, \mathbf{x}_l]) \end{aligned}$$

and since

$$\sharp(\mathbf{N}^d \cap n(0, \mathbf{x}]) = \sharp(\mathbf{N}^{d_1} \cap n(0, \mathbf{x}_1]) \cdot \sharp(\mathbf{N}^{d_2} \cap n(0, \mathbf{x}_2]) \cdots \sharp(\mathbf{N}^{d_l} \cap n(0, \mathbf{x}_l]),$$

we have, by the classical strong law of large numbers,

$$\frac{S_n((0.\mathbf{x}])}{n^d} = \frac{\sharp(\mathbf{N}^d \cap n(0,\mathbf{x}])}{n^d}$$
$$\times \frac{S_{1n}(n(0,\mathbf{x}_1])S_{2n}(n(0,\mathbf{x}_2])\cdots S_{ln}(n(0,\mathbf{x}_l])}{\sharp(\mathbf{N}^{d_1}\cap n(0,\ x_1])\sharp(\mathbf{N}^{d_2}\cap n(0,\mathbf{x}_2])\cdots\sharp(\mathbf{N}^{d_l}\cap n(0,\mathbf{x}_l])}$$
$$\longrightarrow |(0,\mathbf{x}]|\mu_1\mu_2\cdots\mu_l \qquad \text{a.s.},$$

as $n \to \infty$.

But, since any rectilinear set can be obtained by a finite number of unions and differences of rectangles of the form $(0,\ x]$, by linearity we have

$$n^{-d}S_n(A) \longrightarrow \mu_1\mu_2\cdots\mu_l|A| \qquad \text{a.s.},$$

as $n \to \infty$.

Proof of theorem 4.1. The proof is quite similar to the case of product Poisson measures. Write $\mu = \mu_1\mu_2\cdots\mu_l$.

$$\begin{aligned}\limsup_{n\to\infty, A\in\mathcal{A}} \left|n^{-d}S_n(A) - \mu|A|\right| &= \limsup_{n\to\infty, A\in\mathcal{A}} n^{-d}\left|S_n(A) - S_n(R_m^-(A))\right| \\ &+ \limsup_{n\to\infty, A\in\mathcal{A}} \left|n - dS_n(R_m^-(A)) - \mu|R_m^-(A)|\right| \\ &+ \limsup_{n\to\infty, A\in\mathcal{A}} \mu|A \setminus R_m^-(A)| \\ &= T_1 + T_2 + T_3.\end{aligned}$$

Clearly $T_3 \le \mu r(d^{1/2}/m)$ since $\sharp\mathcal{R}_m^- < \infty$. Also

$$\begin{aligned}T_2 &\le \limsup_{n\to\infty, B\in\mathcal{R}_m^-} \left|n^{-d}S_n B) - \mu|B|\right| \\ &\le \limsup_{n\to\infty} \max_{B\in\mathcal{R}_m^-} \left|n^{-d}S(B) - \mu|B|\right| = 0 \ \text{ a.s.}\end{aligned}$$

Finally, let $\alpha_j = E|X_j|$ and $\alpha = \Pi_{j=1}^{j=l}\alpha_j$. For $C \subset \mathbf{I}^d$, set

$$T_n(C) := \sum_{|\mathbf{i}_j|\le n, j=1,2,\cdots,l} |X_{\mathbf{i}_1}||X_{\mathbf{i}_2}|\cdots|X_{\mathbf{i}_l}|\delta_{(\mathbf{i}_1/n,\mathbf{i}_2/n,\cdots,\mathbf{i}_l/n)}(C).$$

By lemma 4.2 applied to the process T_n,

$$T_n \le \limsup_{n\to\infty, A\in\mathcal{A}} n^{-d} T_n(R_m^+(A)\setminus R_m^-(A)) \le \limsup_{n\to\infty} \max_{B\in\mathcal{R}_m^{\triangle}} \left|n^{-d}T_n(B)\right|$$
$$\le \alpha \max_{B\in\mathcal{R}_m^{\triangle}} |B| \le \alpha r(d^{1/2}/m) \qquad \text{a.s.}$$

Summing up and letting $m\to\infty$, we have the conclusion.

References

1. Alexander, K. and Pyke, R., *A uniform limit theorem for set-indexed partial-sum processes with finite variance.*, Ann. Prob **2** (1986), 582-597.
2. Bass, R. F. and Pyke, R., *A strong law of large numbers for partial-sum processes indexed by sets*, Ann. Prob **12** (1984), 268-271.
3. Dudley, R. M., *Metric entropy of some classes of sets with differential boundaries*, J. Approx. Theory **10** (1974), 227-236.
4. Giné, E. and Zinn, J., *The law of large numbers for partial sum processes indexed by sets*, Ann. Prob **15** (1987), 154-163.
5. Hsu, P.L. and Robbins, H., *Complete convergence and the law of large numbers*, Proc. Nat. Aca. Sci **33** (1947), 25-31.
6. Pyke, R., *Asymptotic results for empirical and partial-sum processes: A review*, Canad. J. of Stat **12** (1984), 241-264.
7. Pyke, R., *A century's predictor of future directions*, Preprint, 1991, pp. 37.

DEPARTMENT OF MATHEMATICS, SUN MOON UNIVERSITY, ASAN 336-840, KOREA