APPROXIMATION BY HOLOMORPHIC FUNCTIONS ON PSEUDOCONVEX COMPLEX MANIFOLDS

JINKEE LEE AND HONG RAE CHO

1. Introduction

The following classical Oka-Weil approximation theorem on pseudoconvex domains in $\mathbb{C}^n$ is well-known. Suppose that $M \subseteq \mathbb{C}^n$ is pseudoconvex and that $K$ is a compact subset of $M$ with $\hat{K} = K$, where $\hat{K}$ is the usual holomorphic hull of $K$ in $M$. Then any function holomorphic in a neighborhood of $K$ can be approximated uniformly on $K$ by functions holomorphic on $M$ (see [5], [6]).

Let $X$ be a complex manifold of dimension $n$. Let $M \subseteq X$ be an open pseudoconvex submanifold with smooth boundary. We denote by $H(M)$ the set of functions holomorphic in $M$. Let $P^\infty(M) = P(M) \cap C^\infty(M)$, where $P(M)$ is the set of plurisubharmonic functions on $M$. For a subset $K$ of $\overline{M}$, we define

$$\hat{K}^\infty_P = \{ z \in \overline{M} : \varphi(z) \leq \sup_K \varphi \text{ for all } \varphi \in P^\infty(M) \}.$$

In [1], Catlin showed the following result: If $\hat{K}^\infty_P = K$, then any function holomorphic in a neighborhood of $K$ in the relative topology of $\overline{M}$ can be approximated uniformly on $K$ by functions in $H(M)$. In this paper, we will prove the following theorem:

**Theorem.** Assume that there exists a function $\varphi \in C^\infty(\overline{M})$ which is strongly plurisubharmonic in a neighborhood of $bM$. Let $K$ be a...

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compact subset of $M$ such that $\hat{K} \varphi = K$. If $f$ is a holomorphic function defined on a neighborhood of $K$, then there exists a sequence of holomorphic functions $f_n$ on $M$ such that

$$\lim_{n \to \infty} \sup_K |f_n - f| = 0.$$ 

The Catlin proof involves the $\overline{\partial}$-Neumann problem which contains the difficult questions of regularity at the boundary. But we will use the elementary $\overline{\partial}$-estimates introduced by Hörmander [4]. This method avoids the difficult questions of regularity at the boundary.

2. Preliminaries

Let $\varphi \in C^\infty(M)$ be a real-valued function. We define $L^2_{(0,q)}(M, \varphi)$ as the weighted $L^2$ space of type $(0,q)$ with weight $e^{-\varphi}$. The operator $\overline{\partial}$ defines, in the weak sense, closed densely defined operators

$$L^2_{(0,0)}(M, \varphi) \xrightarrow{T} L^2_{(0,1)}(M, \varphi) \xrightarrow{S} L^2_{(0,2)}(M, \varphi).$$

By $T^*$ we shall mean the adjoint of $T$ with respect to the weighted $L^2$ norm. Then we get the following proposition, its proof is quite similar to that of Hörmander [4].

**Proposition 2.1.** Let $\varphi \in C^\infty(M)$ be a function such that $M_\mu = \{z \in M; \varphi(z) < \mu\} \subseteq M$ and strongly plurisubharmonic in the set $\overline{M} - M_\mu$ for some $\mu \in \mathbb{R}$. Then there exist a compact subset $K$ in $M_\mu$ and a constant $C$ such that for all convex increasing functions $\chi \in C^\infty(\mathbb{R})$

$$(2.1) \quad \int \chi'(\varphi)|f|^2 e^{-\chi(\varphi)}dV \leq C(\|Sf\|_{\chi(\varphi)}^2 + \|T^*f\|_{\chi(\varphi)}^2 + \|f\|_{\chi(\varphi)}^2)$$

for all $f \in \text{Dom}(T^*) \cap \text{Dom}(S)$ with support in $\overline{M} - K$.

Under the hypotheses of Proposition 2.1, we can obtain the following holomorphic approximation theorem which is due to Hörmander ([4, Theorem 3.4.7]). The proof essentially depends on the estimate (2.1).
PROPOSITION 2.2. For every $f \in L^2(M_\mu, \varphi) \cap H(M_\mu)$ and for every $\epsilon > 0$ one can find $f_1 \in L^2(M, \varphi) \cap H(M)$ such that

$$\int_{M_\mu} |f - f_1|^2 e^{-\varphi} dV < \epsilon.$$ 

3. Proof of Theorem

Our manifold may contain a nontrivial compact analytic variety, and on such a variety there cannot exist strongly plurisubharmonic functions. But Catlin constructed a sequence of plurisubharmonic functions on $M$ which would allow us to modify the given plurisubharmonic function on the analytic variety.

**Lemma 3.1.** ([1, Proposition 3.2.1]) There exists a compact analytic variety $S \subset M$ (possibly empty), and a sequence of functions $\lambda_j \in P^\infty(M)$ with the following properties:

1. $\lim_{j \to \infty} \lambda_j(z) = 0$ for $z \notin S$.
2. $\lim_{j \to \infty} \lambda_j(z) = -\infty$ for $z \in S$.
3. For all $j$, $\lambda_j$ is strongly plurisubharmonic off $S$.

Also, we need a technical lemma.

**Lemma 3.2.** Let $K$ be a compact subset of $M$. Let $U$ be a neighborhood of $K$. If $\hat{K}^{\varphi} = K$, then there is $\varphi \in P^\infty(M)$ such that $\varphi < 0$ on $K$ and $\varphi > 0$ on $\overline{M} - U$.

**Proof.** Let $\varphi_0$ be a function in $P^\infty(M)$ such that $\varphi_0 < 0$ on $K$. By the definition of $\hat{K}^{\varphi}$, for each $w \in \overline{M} - U$, there exists $\varphi_w \in P^\infty(M)$ such that $\varphi_w < 0$ on $K$ and $\varphi_w(w) > 0$. Let $\chi \in C^\infty(\mathbb{R})$ be a convex increasing function defined by $\chi(t) = 0$ when $t \leq 0$ and $\chi(t) > 0$ when $t > 0$. By compactness of $\overline{M} - U$, using the Borel-Lebesgue lemma, we can choose points $w_1, \ldots, w_m \in \overline{M} - U$ such that $\sum_{j=1}^m \chi(\varphi_{w_j}) > 0$ on $\overline{M} - U$. Set

$$\varphi = \varphi_0 + C \sum_{j=1}^m \chi(\varphi_{w_j})$$

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where $C$ is chosen so large that $\varphi > 0$ on $\overline{M} - U$. Then $\varphi \in P^\infty(M)$ satisfies that $\varphi < 0$ on $K$ and $\varphi > 0$ on $\overline{M} - U$.

Now we are ready to prove Theorem.

Proof of Theorem. Let $S \subset M$ (possibly empty) be the compact analytic variety, with disjoint connected components $V_1, \ldots, V_N$. Let $f$ be a holomorphic function defined on a neighborhood $U$ of $K$. Since $K_\infty = K$, it follows by the maximum property for plurisubharmonic functions that either $V_j \cap K = \emptyset$ or $V_j \subset K$. Perhaps after shrinking $U$, we may assume that if $V_j \cap K = \emptyset$, then $V_j \cap \overline{U} = \emptyset$. By Lemma 3.2, there exists $\varphi \in P^\infty(M)$ such that $\varphi(z) < 0$ for $z \in K$, and $\varphi > 0$ on $\overline{M} - U$. Let $\{\lambda_j\} \subset P^\infty(M)$ be a sequence as in Lemma 3.1. Set $\varphi_j = \varphi + \lambda_j$. Then for sufficiently large $j$, the function $\varphi_j$ will satisfy the following:

1. $K \subseteq \{z \in \overline{M} : \varphi_j(z) < 0\} \subseteq U$.
2. $\varphi_j$ is strongly plurisubharmonic on $\overline{M} - \{z \in \overline{M} : \varphi_j(z) < 0\}$.

We write the function $\varphi_j$ as $\varphi$ for convenience and then $M_0 = \{z \in \overline{M} : \varphi(z) < 0\}$. Since $\overline{M_0} \subset U$, $f \in L^2(M_0, \varphi) \cap H(M_0)$. By Proposition 2.2, there exists a sequence $\{f_n\} \subset L^2(M, \varphi) \cap H(M)$ such that $f_n \rightarrow f$ in $L^2(M_0, \varphi)$. Since $\varphi \in C^\infty(M)$, the weighted $L^2$ norm is equivalent to the usual $L^2$ norm. Thus $f_n \rightarrow f$ uniformly on compact subsets of the interior of $M_0$ and $K \subseteq M_0$. Thus we complete the proof. □

References

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DEPARTMENT OF MATHEMATICS, PUSAN NATIONAL UNIVERSITY, PUSAN 609-735, KOREA