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APPROXIMATION BY HOLOMORPHIC FUNCTIONS ON PSEUDOCONVEX COMPLEX MANIFOLDS

JINKEE LEE AND HONG RAE CHO

1. Introduction

The following classical Oka-Weil approximation theorem on pseudoconvex domains in $\mathbb{C}^n$ is well-known. Suppose that $M \subseteq \mathbb{C}^n$ is pseudoconvex and that K is a compact subset of M with $\hat{K} = K$, where $\hat{K}$ is the usual holomorphic hull of K in M. Then any function holomorphic in a neighborhood of K can be approximated uniformly on K by functions holomorphic on M (see [5], [6]).

Let X be a complex manifold of dimension n. Let $M \Subset X$ be an open pseudoconvex submanifold with smooth boundary. We denote by $H(M)$ the set of functions holomorphic in M. Let $P^\infty(M) = P(M) \cap C^\infty(\overline{M})$, where $P(M)$ is the set of plurisubharmonic functions on M. For a subset K of $\overline{M}$, we define

$$\hat{K}_P^\infty = \{z \in \overline{M} : \varphi(z) \le \sup_K \varphi \quad \text{for all } \varphi \in P^\infty(M)\}.$$

In [1], Catlin showed the following result: If $\hat{K}_P^\infty = K$, then any function holomorphic in a neighborhood of K in the relative topology of $\overline{M}$ can be approximated uniformly on K by functions in $H(M)$. In this paper, we will prove the following theorem:

THEOREM. *Assume that there exists a function $\varphi \in C^\infty(\overline{M})$ which is strongly plurisubharmonic in a neighborhood of bM. Let K be a*

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compact subset of M such that $\hat{K}_P^\infty = K$. If f is a holomorphic function defined on a neighborhood of K, then there exists a sequence of holomorphic functions f_n on M such that

$$\lim_{n\to\infty} \sup_K |f_n - f| = 0.$$

The Catlin proof involves the $\overline{\partial}$-Neumann problem which contains the difficult questions of regularity at the boundary. But we will use the elementary $\overline{\partial}$-estimates introduced by Hörmander [4]. This method avoids the difficult questions of regularity at the boundary.

2. Preliminaries

Let $\varphi \in C^\infty(\overline{M})$ be a real-valued function. We define $L^2_{(0,q)}(M,\varphi)$ as the weighted L^2 space of type $(0,q)$ with weight $e^{-\varphi}$. The operator $\overline{\partial}$ defines, in the weak sense, closed densely defined operators

$$L^2_{(0,0)}(M,\varphi) \xrightarrow{T} L^2_{(0,1)}(M,\varphi) \xrightarrow{S} L^2_{(0,2)}(M,\varphi).$$

By T^* we shall mean the adjoint of T with respect to the weighted L^2 norm. Then we get the following proposition, its proof is quite similar to that of Hörmander [4].

PROPOSITION 2.1. *Let $\varphi \in C^\infty(\overline{M})$ be a function such that $M_\mu = \{z \in \overline{M}; \varphi(z) < \mu\} \Subset M$ and strongly plurisubharmonic in the set $\overline{M} - M_\mu$ for some $\mu \in \mathbb{R}$. Then there exist a compact subset K in M_μ and a constant C such that for all convex increasing functions $\chi \in C^\infty(\mathbb{R})$*

$$\int \chi'(\varphi)|f|^2 e^{-\chi(\varphi)} dV \leq C(\|Sf\|^2_{\chi(\varphi)} + \|T^*f\|^2_{\chi(\varphi)} + \|f\|^2_{\chi(\varphi)}) \tag{2.1}$$

for all $f \in Dom(T^*) \cap Dom(S)$ with support in $\overline{M} - K$.

Under the hypotheses of Proposition 2.1, we can obtain the following holomorphic approximation theorem which is due to Hörmander ([4, Theorem 3.4.7]). The proof essentially depends on the estimate (2.1).

PROPOSITION 2.2. *For every* $f \in L^2(M_\mu, \varphi) \cap H(M_\mu)$ *and for every* $\epsilon > 0$ *one can find* $f_1 \in L^2(M, \varphi) \cap H(M)$ *such that*

$$\int_{M_\mu} |f - f_1|^2 e^{-\varphi} dV < \epsilon.$$

3. Proof of Theorem

Our manifold may contain a nontrivial compact analytic variety, and on such a variety there cannot exist strongly plurisubharmonic functions. But Catlin constructed a sequence of plurisubharmonic functions on M which would allow us to modify the given plurisubharmonic function on the analytic variety.

LEMMA 3.1. ([1, Proposition 3.2.1]) *There exists a compact analytic variety* $S \Subset M$ *(possibly empty), and a sequence of functions* $\lambda_j \in P^\infty(M)$ *with the following properties:*

(1) $\lim_{j\to\infty} \lambda_j(z) = 0$ *for* $z \notin S$.
(2) $\lim_{j\to\infty} \lambda_j(z) = -\infty$ *for* $z \in S$.
(3) *For all* j, λ_j *is strongly plurisubharmonic off* S.

Also, we need a technical lemma.

LEMMA 3.2. *Let* K *be a compact subset of* M. *Let* U *be a neighborhood of* K. *If* $\hat{K}_P^\infty = K$, *then there is* $\varphi \in P^\infty(M)$ *such that* $\varphi < 0$ *on* K *and* $\varphi > 0$ *on* $\overline{M} - U$.

Proof. Let φ_0 be a function in $P^\infty(M)$ such that $\varphi_0 < 0$ on K. By the definition of $\hat{K}_P^\infty$, for each $w \in \overline{M} - U$, there exists $\varphi_w \in P^\infty(M)$ such that $\varphi_w < 0$ on K and $\varphi_w(w) > 0$. Let $\chi \in C^\infty(\mathbb{R})$ be a convex increasing function defined by $\chi(t) = 0$ when $t \leq 0$ and $\chi(t) > 0$ when $t > 0$. By compactness of $\overline{M} - U$, using the Borel-Lebesgue lemma, we can choose points $w_1, \ldots, w_m \in \overline{M} - U$ such that $\sum_{j=1}^m \chi(\varphi_{w_j}) > 0$ on $\overline{M} - U$. Set

$$\varphi = \varphi_0 + C \sum_{j=1}^m \chi(\varphi_{w_j})$$

where C is chosen so large that $\varphi > 0$ on $\overline{M} - U$. Then $\varphi \in P^{\infty}(M)$ satisfies that $\varphi < 0$ n K and $\varphi > 0$ on $\overline{M} - U$.

Now we are ready to prove Theorem.

Proof of Theorem. Let $S \Subset M$ (possibly empty) be the compact analyic variety, with disjoint connected components $V_1, \ldots, V_N$. Let f be a holomorphic function defined on a neighborhood U of K. Since $\hat{K}_P^{\infty} = K$, if follows by the maximum property for plurisubharmonic functions that either $V_j \cap K = \emptyset$ or $V_j \subset K$. Perhaps after shrinking U, we may assume that if $V_j \cap K = \emptyset$, then $V_j \cap \overline{U} = \emptyset$. By Lemma 3.2, there exists $\varphi \in P^{\infty}(M)$ such that $\varphi(z) < 0$ for $z \in K$, and $\varphi > 0$ on $\overline{M} - U$. Let $\{\lambda_j\} \subset P^{\infty}(M)$ be a sequence as in Lemma 3.1. Set $\varphi_j = \varphi + \lambda_j$. Then for sufficiently large j, the function φ_j will satisfy the following:

(1) $K \Subset \{z \in \overline{M} : \varphi_j(z) < 0\} \Subset U$.
(2) φ_j is strongly plurisubharmonic on $\overline{M} - \{z \in \overline{M} : \varphi_j(z) < 0\}$.

We write the function φ_j as φ for convenience and then $M_0 = \{z \in \overline{M} : \varphi(z) < 0\}$. Since $\overline{M}_0 \subset U, f \in L^2(M_0, \varphi) \cap H(M_0)$. By Proposition 2.2, there exists a sequence $\{f_n\} \subset L^2(M, \varphi) \cap H(M)$such that $f_n \to f$ in $L^2(M_0, \varphi)$. Since $\varphi \in C^{\infty}(\overline{M})$, the weighted L^2 norm is equivalent to the usual L^2 norm. Thus $f_n \to f$ uniformly on compact subsets of the interior of M_0 and $K \Subset M_0$. Thus we complete the proof. □

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DEPARTMENT OF MATHEMATICS, PUSAN NATIONAL UNIVERSITY, PUSAN 609-735, KOREA