Bull. Korean Math. Soc. **32** (1995), No. 2, pp. 265–270

SOME EQUIDIMENSIONAL HILBERT RINGS

HEISOOK LEE AND Y.S. NAM

1. Introduction

Let $K_1, ..., K_n$ be fields of transcendence degrees $t_1, ..., t_n$ respectively over a common subfield F. O'Carroll and Qureshi [7] conjectured that the tensor product $R = K_1 \otimes K_2 \otimes ... \otimes K_n$ is an equidimensional Hilbert ring and proved the conjecture in special cases. Trung proved the conjecture [9] and O'Carroll, Bowman and Howie [3,5] generalized the Trung's result in two directions and obtained two theorems stated below.

THEOREM 1. ([5]) *Let D be a commutative domain, and let $\{B_i | i \in I\}$ be a nonempty collection of subdomains of D, such that :*

(i) *$B = \cap_i B_i$ is infinite, of cardinality $Card(B) > Card(I)$, or B and I are both finite ;*

(ii) *D is a finitely generated B-algebra.*

Let S be the multiplicatively closed set generated by $\cup_i(B_i \backslash 0)$. Then $S^{-1}D$ is a noetherian equidimensional Hilbert ring, of dimension

$$d = min_i\{tr.d.(D/B_i)\}$$

THEOREM 2. ([5], [3]) *Let D be a commutative domain, and let $B_1, ..., B_n$ be subdomains of D ($n > 1$). Suppose that B_1 is chosen so as to satisfy $tr.d.(D/B_1) = min_i\{tr.d.(D/B_i)\}$.*

Suppose further that :

(i) *D is a finitely generated B_i-algebra, $1 < i < n$;*

(ii) *$B_2, ..., B_n$ are integrally closed.*

Received March 22, 1994.
1991 AMS Subject Classification: 13B30.
Key words: Hilbert rings, equidimension.
This work was partially supported by KOSEF.

Let S be the multiplicatively closed set generated by $\cup_i(B_i\backslash 0)$. Then $S^{-1}D$ is a noetherian equidimensional Hilbert ring of dimension

$$d = tr.d.(D/B_1)$$

O'Carroll and Howie ([5], §3, Remark 4) remarked that it can be deduced, from the two theorems, variants to cover the situation where we no longer suppose D to be a domain, but suppose only that each B_i is a domain. However, this remark does not hold in general. We show that $S^{-1}D$ is a noetherian Hilbert ring but not equidimensional in general. Throughout this note, rings are commutative rings with unit elements.

2. Equidimensional Hilbert rings

We first recall the following definitions and basic properties.

DEFINITION 1. A ring R is equidimensional if every maximal ideal of R has the same height.

DEFINITION 2. A ring R is said to be a Hilbert ring (or Jacobson ring) if it satisfies one of the following equivalent conditions :

(1) Every prime ideal is an intersection of maximal ideals.
(2) If M is a maximal ideal in $R[X]$, then $M \cap R$ is a maximal ideal
(3) The Jacobson radical of R/P is (0) for every prime ideal P of R.
(4) For any prime ideal P of R and any element a of R with $a \notin P$, there exists a maximal ideal M of R such that $P \subset M$ and $a \notin M$.

Basic properties of Hilbert rings [2]

(1) Any homomorphic image of a Hilbert ring is a Hilbert ring.
(2) A ring R is a Hilbert ring if and only if the polynomial ring $R[X]$ is a Hilbert ring.

From the definition and the above property (1), the following lemmas can be easily obtained.

LEMMA 3. *A ring R is a Hilbert ring if and only if R/P is a Hilbert ring for every minimal prime ideal P of R.*

Proof. By property (1), R/P is a Hilbert ring if R is.

To prove the converse, let Q be a prime ideal of R. Then there exist a minimal prime ideal P of R contained in Q. By the definition of a Hilbert ring, Jacobson radical R/Q is 0. Since the Jacobson radical R/Q is isomorphic to Jacobson radical of $(R/P)/(Q/P)$, R is a Hilbert ring.

LEMMA 4. *Rings A and B are Hilbert rings if and only if $A \oplus B$ is a Hilbert ring.*

Proof. Since the prime ideals of $A \oplus B$ are of the form $P \oplus B$ or $A \oplus Q$, where P and Q are prime ideals of A and B respectively, the lemma is proved by (3) of the definition of a Hilbert ring .

Note : A localization of a Hilbert ring is not necessarily a Hilbert ring. In fact let p be a prime ideal of a Hilbert ring R with height h with $h > 1$. Then R_p is not a Hilbert ring, since a local Hilbert ring must be 0-dimensional.

However there are rings whose localizations are Hilbert rings as in Theorems 1 and 2 in the introduction. More generally we deduce that $S^{-1}D$ is a noetherian Hilbert ring in the situation of Theorem 1 and 2 in case that D is not necessarily a domain.

PROPOSITION 5. *Let D be a commutative ring and let $\{B_i | i \in I\}$ be a nonempty collection of subdomains of D, such that :*

(i) *$B = \cap_i B_i$ is infinite, of cardinality $Card(B) > Card(I)$, or B and I are both finite ;*

(ii) *D is a finitely generated B-algebra.*

Let S be the multiplicatively closed set generated by $\cup_i (B_i \backslash 0)$. Then $S^{-1}D$ is a noetherian Hilbert ring.

Proof. Let $T = B \backslash \{0\}$. Then as in the proof of Theorem 1 [5], $S^{-1}D$ is a localization of a finitely generated algebra $T^{-1}D$ over a field $T^{-1}B$. Hence $S^{-1}D$ is a noetherian ring. To prove $S^{-1}D$ is a Hilbert ring, let X be the set of minimal prime ideals of $S^{-1}D$. Then for any p in X, there exist a prime ideal q in D such that $q \cap B_i = (0)$ for all i and $p = S^{-1}q$. Let $\overline{S}, \overline{B}$ and $\overline{B}_i$ be the images of S, B and B_i respectively in the ring $\overline{D} = D/q$. Then $\overline{S} \simeq S, \overline{B} \simeq B$ and $\overline{B}_i \simeq B_i$. Since $\overline{D}, \overline{S}, \overline{B}$, and $\overline{B}_i$ satisfy the conditions in Theorem 1, $S^{-1}\overline{D}$ is a Hilbert ring. Hence $S^{-1}D$ is a Hilbert ring by Lemma 3, since $S^{-1}D/p = \overline{S}^{-1}\overline{D}$.

PROPOSITION 6. *Let D be a commutative ring and let $B_1, ..., B_n$ be subdomains of $D(n > 1)$. Suppose that :*

(i) *D is a finitely generated B_i-algebra, $1 < i < n$;*

(ii) *$B_2, ..., B_n$ are integrally closed.*

Let S be the multiplicatively closed set generated by $\cup_i(B_i \backslash 0)$. Then $S^{-1}D$ is a noetherian Hilbert ring.

Proof. If we take $T = B_1 \backslash \{0\}$, then the same proof in Proposition 5 holds.

It was remarked that $S^{-1}D$ in Proposition 5 and 6 is equidimensional [5]. However the following example shows that $S^{-1}D$ is not equidimensional in general.

EXAMPLE. Let F be a field and let X and Y be independent indeterminants over F. In $F[X, Y]$, let $I = (X)(X-1, Y) = (X(X-1), XY)$.

Consider $D = F[X, Y]/I$. Let B_1 be the image of F in D. Then D and B_1 satisfy the hypothesis of Theorem1 except that D is a domain. Furthermore, every element of $S = B_1 \backslash 0$ is a unit in D and hence $S^{-1}D = D$. Thus $(X, Y)/I$ and $(X-1, Y)/I$ are maximal ideals of $S^{-1}D$, whereas $ht[(X, Y)/I] = 1$ and $ht[(X-1, Y)/I] = 0$. Hence $S^{-1}D$ is not equidimensional, but a noetherin Hilbert ring.

We note that in the above example (X) and $(X-1, Y)$ are minimal prime ideals over I and hence D has minimal prime ideals $P = (X)/I$ and $Q = (X-1, Y)/I$. However, $dim D/P = 1$ but $dim D/Q = 0$.

We now consider the condition that $S^{-1}D$ is equidimensional.

Let D, B and S be as in Proposition 5. Let $T = B\backslash\{0\}$. Then $T^{-1}D$ is a finitely generated algebra over field $T^{-1}B$. Then the dimension of $T^{-1}D$ is finite and hence the dimension of $S^{-1}D$ is finite. By replacing B by $T^{-1}B$ and D by $T^{-1}D$, we may assume that D and B contain a field and are of finite dimension.

PROPOSITION 7. *Let D be a commutative ring and let $\{B_i | i \in I\}$, B and S be as in Proposition 5. We assume that for all prime ideals which are minimal with the property that $P \cap B_i = \{0\}$ for all i, $dim D/P$ are equal to h. Then $S^{-1}D$ is an equidimensional ring of dimension $d = h - max_i\{dim B_i\}$.*

Proof. Let $P_0 \subset P_1 \subset ... \subset P_c$ be a saturated chain of prime ideals in $S^{-1}D$ with P_0 a minimal ideal and P_c a maximal ideal. Then c is finite since $dimS^{-1}D$ is finite. Since P_0 is a minimal prime ideal, $P_0 = S^{-1}p$ for some prime ideal p of D which is minimal with the property that $p \cap B_i = \{0\}$ for all i.

Let $\overline{D} = D/p$ and let $\overline{S}, \overline{B}$ and $\overline{B}_i$ denote the images of S, B and B_i in $\overline{D}$ respectively. Then $(S^{-1}D)/P_0 = (\overline{S})^{-1}(\overline{D})$ and hence $0 \subset P_1/P_0 \subset ... \subset P_c/P_0$ can be regarded as a saturated chain of prime ideals in $(\overline{S})^{-1}(\overline{D})$ with P_c/P_0 maximal.

Furthermore, $\overline{D}$ is a domain, which is finitely generated as a B-algebra in the situation of Theorem 1. In addition B and B_i are mapped isomorphically onto $\overline{B}$ and $\overline{B}_i$, respectively by the natural map, so $dim\overline{B}_i = dimB_i$, $\overline{B} = \cap_i\overline{B}_i$ and $\overline{S}$ is the multiplicatively closed set generated by $\cup_i(B_i\backslash 0)$. Thus $\overline{D}$ and $\overline{B}_i$ satisfy the hypothesis of Proposition 5, from which it follows that $(\overline{S})^{-1}(\overline{D})$ is an equidimensional Hilbert ring, of dimension

$$dim\overline{D} - max_i\{dimB_i\} = h - max\{dim\overline{B}_i\}.$$

Hence $c = d$. This completes the proof.

Let B_i be an integral domain in a commutative ring D and let $T_i = B_i\backslash\{0\}$. Then $T_i^{-1}D$ is a finitely generated algebra over a field $T_i^{-1}B_i$ and hence $dimT_i^{-1}D$ is finite. Passing to $T_i^{-1}B_i$, we may assume that D is a finitely generated algebra over a field and $dimD$ is finite. Using the similar argument in Proposition 7, we obtained the following.

PROPOSITION 8. *Let $D, \{B_1, ..., B_n\} < B$ and S be as in Proposition 6. We suppose that B_1 is chosen so as to satisfy $dim(T_1^{-1}D) = min_i(T_i^{-1}D)$. If we further assume that $dimD/p$ are equal for all prime ideals p which are minimal having zero intersection with each B_i. Then $S^{-1}D$ is an equidimensional ring of dimension $dim(T_1^{-1}D)$.*

A ring R is said to be a normal ring if R_p is an integrally closed domain for each prime ideal p of R. We recall that a noetherian ring R is normal if and only if R is a finite direct sum of noetherian normal domains.

REMARK. Now we consider the variants of Proposition 6 and 8 as in the Remark 4 [5], i.e., without assuming that $B_2, \ldots, B_n$ are integrally closed domain, we add the new hypothesis that D is a normal ring. Then the same results hold, because of Lemma 4.

References

1. M. F. Atiyah and I. G. Macdonald, *Introduction to Commutate Algebra*, Addision-Wesley, Reading, Mass., 1969.
2. N. Brbaki, *Elements of Mathematics, Commutative Algebra*, Herman, Paris, 1972.
S. R. Bowman and O'Carroll, *Tensor products and localizations of algebras*, Nagoya Math. J. **102** (1986), 155-161.
4. K. Fujita, *A note on Hilbert's Nullstellensatz*, Hiroshima Math. J. **4** (1974), 421-424.
5. J.Howie and L.O'Carroll, *Some localizations which are Hilbert rings*, J. Algebra **92** (1985), 366-374.
6. H. Matsmura, *Commutative Algebra*, Benjamin/Cummings, Reading, Massachusetts, 1980.
7. L. O'Carroll and M. A. Qureshi, *On the tensor product of fields and algebraic correspondences*, Quart. J. Math.(2) **34** (1983), 211-221.
8. R. Y. Sharp and P. Vamos, *The dimension of tensor product of a finite number of field extensions*, J.Pure Appl. Algebra **10** (1977), 249-252.
9. N. V. Trung, *On the tensor product of extensions of a field*, Quart J. Math.(2) **35** (1984), 337-339.
10. P. Vamos, *The Nullstellensatz and Tensor product of fields*, Bull. London Math. Soc. **9** (1977), 273-278.

HEISOOK LEE AND Y.S. NAM
DEPARTMENT OF MATHEMATICS, EWHA WOMANS UNIVERSITY, SEOUL, KOREA