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RANGE OF PARAMETER FOR THE EXISTENCE OF PERIODIC SOLUTIONS OF LIÉNARD DIFFERENTIAL EQUATIONS

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1. Introduction

In 1986, Fabry, Mawhin and Nkashama [1] have considered periodic solutios for Liénard equation

$$(1_s) \qquad x'' + f(x)x' + g(t,x) = s,$$

where s is a real parameter, f and g are continuous functions, and g is 2π-periodic in t and have proved that if

$$(\mathrm{H}) \qquad \lim_{|x|\to\infty} g(t,x) = \infty \quad \text{uniformly in } t \in [0, 2\pi],$$

there exists $s_1 \in \mathbf{R}$ such that (1_s) has no 2πperiodic solution if $s < s_1$, and at least one 2π-periodic solution if $s = s_1$, and at least two 2π-periodic solutions if $s > s_1$.

Problems (Results) depending on a real parameter s and having a similar result like the above are called the Ambrosetti-Prodi type problems (results) and this type of problems has been studied for a wide class of differential equations ([2] and references therein).

What we are concerned here is range of parameter s_1 of the existence for which one may intuitively guess that s_1 may strongly interact with the nonlinear term. In this note, we prove a little weak version of

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Ambrosetti-Prodi type result for the problem and give a bound of s_1 in terms of $g(t,x)$. The proof is basically different from [1] and generally along the lines of [2].

In what follows, $J = [0, 2\pi]$. Mean value $\bar{x}$ of x and the function $\tilde{x}$ of mean value 0 are respectively defined by $\bar{x} = \frac{1}{2\pi}\int_0^{2\pi} x(t)dt$ and $\tilde{x}(t) = x(t) - \bar{x}$. The symbol $C^k(J)$ will be denote the real Banach space of continuous functions $J \mapsto \mathbf{R}$ whose derivatives through order k are also continuous, and $C^k_{2\pi}(J)$ the real Banach space of 2π-periodic functions of class $C^k(J)$. Both spaces will be equipped with norm $\| x \|_\infty + \cdots + \| x^{(k)} \|_\infty$, where $\| u \|_\infty = \sup_{t\in J} |u(t)|$. If $x \in C^0_{2\pi}(J)$ we define $\| x \|_2^2 = \frac{1}{2\pi}\int_0^{2\pi} |x(t)|^2 dt$.

SOBOLEV INEQUALITY [3]. If $x \in C^0_{2\pi}(J) \cap C^1(J)$ and of mean value 0, then

$$\| x \|_\infty \le \frac{\pi}{\sqrt{3}} \| x' \|_2 \le \frac{\pi}{\sqrt{3}} \| x' \|_\infty .$$

By the continuity of g and condition (H), g attains the minimum, so let

$$g(t_o, x_o) = \min_{\substack{t\in J \\ x\in\mathbf{R}}} g(t,x).$$

Now we state the main theorem.

THEOREM 1. *If (H) is satisfied, and let $g(t_o, x_o) = \min_{t,x} g(t,x)$, then for $s_o = \min_{t,x} g(t,x)$ and $\bar{s} = \max_{t\in J} g(t, x_o)$,*

(i) *$({}_s)$ has no 2π-periodic solution for $s < s_o$,*
(ii) *(1_s) has at least one 2π-periodic solution for $s = \bar{s}$,*
(iii) *(1_s) has at least two 2π-periodic solution for $s > \bar{s}$.*

We give *a priori* bound of possible 2π−periodic solutions for a homotopy of (1_s) in section 2, and some degree computations and proof of the main theorem in section 3, and we end up with some remarks on the range of s_1 and an autonomous Liénard differential equation.

2. A priori estimate

For convinience, we consider the problem under the following condition for a while.

$$\text{(C)} \qquad 0 = g(t_o, 0) = \min_{\substack{t \in J \\ x \in \mathbf{R}}} g(t, x)$$

We shall obtain *a priori* bound for possible 2π-periodic solutions of

$$(2_s^\mu) \qquad x''(t) + (1-\mu)|x(t)| + \mu f(x(t))x'(t) + \mu g(t, x(t)) = s,$$

where $\mu \in [0,1]$.

LEMMA 1. *If (H) is satisfied, then for each $s^* \in \mathbf{R}$, there exist M, $V > 0$ such that for each $s \leq s^*$ and each possible 2π-periodic solution x of (2_s^μ), one has*

$$\| x \|_\infty < M, \quad \| x' \|_\infty <$$

Proof. Let s^* be given, $s \leq s^*$ and let x be a 2π-periodic solution of (2_s^μ), then

$$(3) \qquad \frac{1}{2\pi}\int_0^{2\pi} (1-\mu)|x(t)| + \mu g(t, x(t))dt = s.$$

Without loss of generality, we assume $0 \leq s$ $(\leq s^*)$, otherwise (2_s^μ) does not have a 2π-periodic solution by (3). Multiplying $\tilde{x}$ and integrating both sides of (2_s^μ), we get

$$\begin{aligned}
\|x'\|_2^2 &= \frac{1}{2\pi}\int_0^{2\pi} \tilde{x}(t)\{(1-\mu)|x(t)| + \mu g(t, x(t))\}dt \\
&\leq \frac{1}{2\pi}\|\tilde{x}\|_\infty \int_o^{2\pi} (1-\mu)|x(t)| + \mu g(t, x(t))dt \\
&= s\|\tilde{x}\|_\infty \\
&\leq \frac{\pi}{\sqrt{3}} s^* \|x'\|_2, \quad \text{by Sobolev inequality.}
\end{aligned}$$

Therefore

$$\|x'\|_2 \le \frac{\pi}{\sqrt{3}} s^*.$$

Next, we claim that $x(\tau)$ is bounded for some $\tau \in J$ and for all possible solutions x. We may assume by (H) that there exists $r(s^*) > 0$ such that

$$(4) \qquad g(t,x) > s^*, \quad \text{whenever } |x| \ge r(s^*)$$

for all $t \in J$. Mean value theorem for (3) provides that there exists $\tau \in J$ such that

$$(1-\mu)|x(\tau)| + \mu g(\tau, x(\tau)) = s.$$

This implies that s lies between $|x(\tau)|$ and $g(\tau, x(\tau))$, and thus we have either

$$(5) \qquad |x(\tau)| \le s \quad \text{or} \quad g(\tau, x(\tau)) \le s.$$

We show that $|x(\tau)| < r(s^*)$ if $g(\tau, x(\tau)) \le s$. Suppose $|x(\tau)| \ge r(s^*)$, then by (4), $g(\tau, x(\tau)) > s^*$. Thus

$$s \le s^* < g(\tau, x(\tau)) \le s.$$

This contradiction shows that

$$(6) \qquad |x(\tau)| < r(s^*).$$

By (5) and (6),

$$|x(\tau)| \le \max\{s^*, r(s^*)\}.$$

Choosing R strictly greater than $\max\{s^*, r(s^*)\}$, we get

$$|x(\tau)| < R.$$

Finally,

$$\begin{aligned} |x(t)| &\le |x(\tau)| + \left| \int_\tau^t x'(s)ds \right| \\ &\le |x(\tau)| + 2\pi \|x'\|_2 \\ &< R + \frac{2\pi^2}{\sqrt{3}} s^* \equiv M, \end{aligned}$$

for all $t \in J$. Therefore

$$\|x\|_\infty < M.$$

Since $\|x'\|_2$ is bounded, it is not hard to show in (2_s^μ) that $\|x''\|_2$ is bounded, say $\|x''\|_2 < \frac{\sqrt{3}}{\pi}V$ for some constant V. Thus by the Sobolev inequality, we obtain

$$\|x'\|_\infty < V.$$

3. Degree computations

We reduce problem (2_s^μ) to an equivalent operator form. Let us define $L : D(L) \subset C_{2\pi}^o(J) \longrightarrow C^0(J)$ by $x \mapsto x''$, where $D(L) = C_{2\pi}^2(J)$, and for each $\mu \in [0,1]$, $N_s^\mu : C_{2\pi}^1(J) \longrightarrow C^0(J)$ by

$$N_s^\mu x(\cdot) = (1-\mu)|x(\cdot)| + \mu f(x(\cdot))x'(\cdot) + g(\cdot, x(\cdot)) - s$$

so that (2_s^μ) and (1_s) can be written

$$Lx + N_s^\mu x = 0,$$
$$Lx + N_s^1 x = 0$$

respectively. It is well-known that L is a Fredholm operator with index 0 and N_s^μ is L-compact on $\overline{\Omega}$ for any open bounded $\Omega \subset C_{2\pi}^0$. The coincidence degree $D_L(L + N_s^\mu, \Omega)$ is well-defined and constant in μ if $L + N_s^\mu(x) \neq 0$ for $\mu \in [0,1]$, $s \in \mathbf{R}$ and $x \in D(L) \cap \partial\Omega$.

LEMMA 2. *If (H) is satisfied, then for each $s^* \geq 0$ and for each open bounded set $\Omega(s^*) \subset C_{2\pi}^1(J)$ such that*

$$\Omega(s^*) \supset \{x \in C_{2\pi}^1(J) : \|x\|_\infty < M,\ \|x'\|_\infty < V\},$$

one has

$$D_L(L + N_s^1, \Omega(s^*)) = 0 \quad \textit{whenever } s \leq s^*.$$

Proof. Let s^* be given and let $\Omega(s^*)$ be any subset of $C_{2\pi}^1$ containing $\{x \in C_{2\pi}^1(J) : \|x\|_\infty < M,\ \|x'\|_\infty < V\}$. Let $s_o = \min_{\substack{t \in J \\ x \in \mathbf{R}}} g(t,x)\ (=0)$. If (1_s) has a 2π-periodic solution x, then

$$s_o \leq \frac{1}{2\pi}\int_0^{2\pi} g(t, x(t))dt = s.$$

This implies that (1_s) has no 2π-periodic solution for $s < s_o$. Thus by the existence property of degree, we have

$$D_L(L + N_s^1, \Omega(s^*)) = 0 \quad \text{for } s < s_o.$$

By *a priori* estimate and the homotopy invariance of degree, we have, for fixed $\tilde{s} < s_o$,

$$D_L(L + N_s^1, \Omega(s^*)) = D_L(L + N_{\tilde{s}}^1, \Omega(s^*)) = 0,$$

for all $s \leq s^*$.

LEMMA 3. *If (H) is satisfied, then there exists $\bar{s} \geq s_o$ such that for each $s^* > \bar{s}$, one can find an open bounded set $\Omega_1(s^*)$ in $C^1_{2\pi}(J)$ for which*

$$|D_L(L + N_s^1, \Omega_1(s^*))| = 1,$$

for $\bar{s} < s \leq s^*$.

Proof. Let

$$\bar{s} = \max_{t \in J} g(t, 0)$$

then $\bar{s} \geq \bar{s}_o$. Let $s^* > \bar{s}$ and let

$$\Omega_1(s^*) = \{x \in C^1_{2\pi}(J) : -M < x(t) < 0, \ t \in J, \ \|x'\|_\infty < V\}$$

then $\Omega_1(s^*) \subset \Omega(s^*)$ and since $x \in \Omega_1(s^*)$ is negative, (2_s^μ) is equivalent to

$$x''(t) - (1 - \mu)x(t) + \mu f(x(t))x'(t) + \mu g(t, x(t)) = s,$$

on $\Omega_1(s^*)$. We show that $D_L(L + N_s^\mu, \Omega(s^*))$ is well-defined for $\bar{s} < s \leq s^*$. For $0 < \mu \leq 1$, let x be a 2π-periodic solution for (2_s^μ) for $x \in \partial\Omega_1(s^*)$, then $x(\tau) = 0$ for some $\tau \in J$. Since $x(\tau) = \max_{t \in J} x(t)$, $x'(\tau) = 0$ and $x''(\tau) \leq 0$. Thus from (2_s^μ),

$$x''(\tau) + \mu g(t, x(\tau)) = s$$

and

$$s \leq \mu g(t, 0) \leq \sup_{t \in J} g(t, 0) = \bar{s} < s.$$

This contradiction proves that the degree is well-defined for $\mu \in (0,1]$. For $\mu = 0$, $x''(t) - x(t) = s$ has the only 2π-periodic solution $x(t) = -s$. Thus for $\bar{s} < s \leq s^*$, $x \in \Omega_1(s^*)$. By the normalization property and homotopy invariance of degree, we get

$$\begin{aligned} 1 &= |D_L(L - I - s, \Omega_1(s^*))| \\ &= |D_L(L + N_s^0, \Omega_1(s^*))| \\ &= |D_L(L + N_s^1, \Omega_1(s^*))|. \end{aligned}$$

The following lemma is a parallel version of the main theorem under condition (C).

LEMMA 4. *If (H) and (C) are satisfied, then for $s_o = \min_{t,x} g(t,x)$ and $\bar{s} = \max_t g(t, x_o)$,*

(i) *(1_s) has no 2π-periodic solution for $s < s_o$,*
(ii) *(1_s) has at least one 2π-periodic solution for $s = \bar{s}$,*
(iii) *(1_s) has at least two 2π-periodic solutions for $s > \bar{s}$.*

Proof. (i) has been proved in Lemma 2. For (iii), if $s > \bar{s}$ then we choose $\Omega(s) \supset \Omega_1(s)$, both of which are defined in Lemma 2 and Lemma 3 respectively, and by the additivity property of degree, we have

$$0 = D_L(L+N_s^1, \Omega(s)) = D_L(L+N_s^1, \Omega_1(s)) + D_L(L+N_s^1, \Omega(s) \setminus \overline{\Omega_1(s)}).$$

Since $|D_L(L + N_s^1, \Omega_1(s))| = 1$ by Lemma 3, we have

$$|D_L(L + N_s^1, \Omega(s) \setminus \overline{\Omega_1(s)})| = 1$$

and thus (1_s) has one 2π-periodic solution in $\Omega(s)$ and another one in $\Omega(s) \setminus \overline{\Omega_1(s)}$. For (ii), let (s_n) be a sequence in $\mathbf{R}$ such that $s_n \to \bar{s}$ and let x_n be a 2π-periodic solution for (1_{s_n}), then by *a priori* estimate, (x_n) is bounded in $C^1_{2\pi}(J)$ with norm $\|x\|_\infty + \|x'\|_\infty$. Since x_n is a solution of (1_{s_n}) and f, g are continuous, (x_n'') is bounded with the sup-norm so that (x_n) is bounded in $C^2_{2\pi}(J)$ with norm $\|x\|_\infty + \|x'\|_\infty + \|x''\|_\infty$. Since $C^2_{2\pi}$ is compactly embedded in $C^1_{2\pi}$, (x_n) has a subsequence converging to x in $C^1_{2\pi}(J)$. Writing equation (1_s) in integral form, one can easily show that the limit x is a solution of $(1_{\bar{s}})$, and this completes the proof.

We now prove the main theorem.

Proof of Theorem 1. Let $g(t_o, x_o) = \min_{t,x} g(t,x)$ and let

$$\begin{aligned}\tilde{g}(t,x) &= g(t, x + x_o) - g(t_o, x_o)\\ \tilde{f}(x) &= f(x + x_o)\\ \tilde{s} &= s - g(t_o, x_o).\end{aligned}$$

Then

$$\tilde{g}(t_o, 0) = 0 \quad \text{and} \quad \tilde{g}(t,x) \geq 0,$$

for all $t \in J$, $x \in \mathbf{R}$ and furthermore,

$$\tilde{g}(t,x) \longrightarrow \infty \text{ uniformly in } t, \quad \text{as } |x| \to \infty.$$

Hence the equation

$$(7_{\tilde{s}}) \qquad x'' + \tilde{f}(x)x' + \tilde{g}(t,x) = \tilde{s}$$

satisfies hypotheses in Lemma 4 so that the conclusion (i) $\sim$ (iii) holds for $\tilde{s_o} = \min_{t,x} \tilde{g}(t,x)$ and $\bar{\tilde{s}} = \max_t g(t, x_o) - g(t_o, x_o)$. We notice that $\tilde{s} < \tilde{s_o}$ and $\tilde{s} > \bar{\tilde{s}}$ are equivalent to $s < s_o = \min_{t,x} g(t.x)$ and $s > \bar{s} = \max_t g(t, x_o)$ respectively. Let y be a 2π-periodic solution of $(7_{\tilde{s}})$ and let $x(t) = y(t) + x_o$, then $(7_{\tilde{s}})$ is equivalent to (1_s). Consequently, (1_s) holds (i) $\sim$ (iii) for $s_o = \min_{t,x} g(t,x)$ and $\bar{s} = \max_t g(t, x_o)$ and the proof is complete.

REMARK 1. The existence parameter s_1 in [1] satisfies that for each $(t_o, x_o) \in J \times \mathbf{R}$ with $g(t_o, x_o) = \min_{t,x} g(t,x)$,

$$\min_{t,x} g(t,x) \leq s_1 \leq \max_t g(t, x_o).$$

Since x_o may occur at infinitely many different places in a bounded subset of $J \times \mathbf{R}$, we conclude

$$\min_{t,x} g(t,x) \leq s_1 \leq \inf_{x_o} \max_t g(t, x_o).$$

REMARK 2. If $g(t,x) = g(x)$, then s_o and $\bar{s}$ in Theorem 1 are the same. Thus autonomous Liénard differential equation

$$(8_s) \qquad x'' + f(x)x' + g(x) = s$$

satisfies that for $s_1 = \min_x g(x)$,

(i) (8_s) has no 2π-periodic solution for $s < s_1$,
(ii) (8_s) has at least one 2π-periodic solution for $s = s_1$,
(iii) (8_s) has at least two 2π-periodic solutions for $s > s_1$.

The result is the same as [1] and moreover, we give a sharp estimate of s_1 while [1] gives only the fact that $s_1 \geq \min_x g(x)$.

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