Bull. Korean Math. Soc. **32** (1995), No. 2, pp. 309–319

CONFORMAL CURVATURE TENSOR FIELD AND SPECTRUM OF THE LAPLACIAN IN KAEHLERIAN MANIFOLDS*

JIN SUK PAK, KWAN-HO CHO AND JUNG-HWAN KWON

1. Introduction

Let (M,g) be an m-dimensional compact orientable Riemannian manifold(connected and C^∞) with metric tensor g. We denote by $\triangle$ the Laplacian acting on p-forms on M, $0 \le p \le m$. Then we have the spectrum for each p :

$$Spec^p(M,g) = \{0 \le \lambda_{0,p} \le \lambda_{1,p} \le \lambda_{2,p} \le \cdots \uparrow +\infty\},$$

where each eigenvalue $\lambda_{\alpha,p}$ is repeated as many times as its multiplicity indicates. In order to study the relation between $Spec^p(M,g)$ and the geometry of (M,g) we use the Minakshisundaram-Pleijel-Gaffney's formula. J. S. Pak, J. C. Jeong and W.-T. Kim[4], S. Yamaguchi and G.Chūman[9] and others studied the spectrum of the Laplacian and the curvature of Sasakian manifolds. J. S. Pak, J.-H. Kwon and K.-H. Cho[5] studied the spectrum of the Laplacian and the curvature of cosymplectic manifolds.

The purpose of the present paper is to study the spectrum of the Laplacian and the conformal curvature tensor field of Kaehlerian manifolds.

Received April 20, 1994.

1991 AMS Subject Classification: 53C55, 53C25.

Key words: conformal curvature tensor field, Laplacian, Einstein, spectrum, Kaehlerian manifold.

*This research was partially supported by TGRC-KOSEF and the Basic Science Research Institute Program, Ministry of Education, 1995. Project No. BSRI-95-1404.

We shall be in C^∞-category. The indices $h, i, j, k, s, t, \ldots$ run over the range $\{1, 2, \ldots, 2n\}$. The Einstein summation convention with respect to those system of indices will be used.

2. Preliminaries

By $R = (R_{kji}{}^h)$, $R_1 = (R_{ji})$ and r we denote the Riemannian curvature tensor, the Ricci curvature tensor and the scalar curvature, respectively.

For a tensor field T on M, we denote by $\|T\|$ the norm of T with respect to g. Then the Minakshisundaram-Pleijel-Gaffney's formula for $Spec^p(M, g)$ is given by

$$\sum_{\alpha=0}^{\infty} \exp(-\lambda_{\alpha,p} t) \sim (4\pi t)^{\frac{-m}{2}} \sum_{\alpha=0}^{\infty} a_{\alpha,p} t^\alpha \text{ as } t \to 0^+,$$

where the constants $a_{\alpha,p}$ are spectral invariants. In the present paper we are interested in the case of $p = 0, 1$ or 2. For $p = 0$, we have (cf.[1])

$$a_{0,0} = \int_M dM = Vol(M, g), \tag{2.1}$$

$$a_{1,0} = \frac{1}{6} \int_M r dM, \tag{2.2}$$

$$a_{2,0} = \frac{1}{360} \int_M \left[2\|R\|^2 - 2\|R_1\|^2 + 5r^2\right] dM, \tag{2.3}$$

where dM denotes the natural volume element of (M, g). For $p = 1$, we have (cf. [9])

$$a_{0,1} = mVol(M, g), \tag{2.4}$$

$$a_{1,1} = \frac{m-6}{6} \int_M r dM, \tag{2.5}$$

$$a_{2,1} = \frac{1}{360}\int_M \left[2(m-15)\|R\|^2 - 2(m-90)\|R_1\|^2 + 5(m-12)r^2\right] dM. \tag{2.6}$$

For $p = 2$, we have (cf.[6], [8], [9])

$$a_{0,2} = \frac{1}{2}m(m-1)Vol(M,g), \tag{2.7}$$

$$a_{1,2} = \frac{1}{12}(m^2 - 13m + 24)\int_M r dM, \tag{2.8}$$

$$a_{2,2} = \frac{1}{720}\int_M \left[2(m^2 - 31m + 240)\|R\|^2 - 2(m^2 - 181m + 1080)\|R_1\|^2 + 5(m^2 - 25m + 120)r^2\right] dM. \tag{2.9}$$

3. Kaehlerian manifolds

Let M be a $2n$-dimensional differentiable Kaehlerian manifold of class C^∞ covered by a system of coordinate neighborhoods $\{U; x^h\}$ and $\varphi_j{}^i$ the almost complex structure tensor and g_{ji} the Hermitian metric tensor. Then we have

$$\varphi_j{}^t\varphi_t{}^i = -\delta_j{}^i, \quad \varphi_j{}^t\varphi_i{}^s g_{ts} = g_{ji}, \tag{3.1}$$

$$\nabla_j \varphi_i{}^h = 0, \tag{3.2}$$

where ∇ denotes the operator of covariant differentiation with respect to the Christoffel symbols formed with g_{ji}. If we define $\varphi_{ji} = \varphi_j{}^t g_{ti}$, we see from (3.1) and (3.2) that φ_{ji} is skew-symmetric.

If we denote the curvature tensor, Ricci tensor and scalar curvature of a Kaehlerian manifold M by $R_{kji}{}^h$, R_{ji} and r respectively, then we have

$$\begin{aligned} &R_{tjih}\varphi_k{}^t = -R_{ktih}\varphi_j{}^t, && R_{tjis}\varphi^{ts} = -R_{jt}\varphi_i{}^t, \\ &R_{tsji}\varphi^{ts} = 2R_{jt}\varphi_i{}^t, && R_{jt}\varphi_i{}^t = -R_{it}\varphi_j{}^t, \\ &R_{jtsh}\varphi^{sh}\varphi_i{}^t = -2R_{ji}, && R_{ts}\varphi_j{}^t\varphi_i{}^s = R_{ji}. \end{aligned} \tag{3.3}$$

where $\varphi^{ji} = \varphi_t{}^i g^{jt}$, $R_{kjih} = R_{kji}{}^t g_{th}$.

A tensor field $Q = (Q_{ji})$ on M is defined by

$$Q_{ji} = R_{ji} - \frac{r}{2n} g_{ji}.$$

By a direct calculation, in which we use (3.3), it follows that

$$\|Q\|^2 = \|R_1\|^2 - \frac{1}{2n} r^2. \tag{3.4}$$

A Kaehlerian manifold is said to be *Einstein* if $Q = 0$. For any Einstein Kaehlerian manifold, r is constant, provided $n \geq 2$.

We also consider the so-called *conformal curvature tensor field* $B_0 = (B_{kjih})$ defined on M by (cf.[2],[3])

$$\begin{aligned} B_{0,kjih} =& R_{kjih} + \frac{1}{2n}(g_{ji}R_{kh} - g_{ki}R_{jh} + R_{ji}g_{kh} - R_{ki}g_{jh} \\ & - \varphi_{ji}S_{kh} + \varphi_{ki}S_{jh} - S_{ji}\varphi_{kh} \\ & + S_{ki}\varphi_{jh} + 2\varphi_{ih}S_{kj} + 2S_{ih}\varphi_{kj}) \\ & + \frac{(n+2)r}{4n^2(n+1)}(\varphi_{ji}\varphi_{kh} - \varphi_{ki}\varphi_{jh} - 2\varphi_{ih}\varphi_{kj}) \\ & - \frac{(3n+2)r}{4n^2(n+1)}(g_{ji}g_{kh} - g_{ki}g_{jh}), \end{aligned} \tag{3.5}$$

where $S_{ji} = -R_{jt}\varphi_i{}^t$ and $S_{ji} = -S_{ij}$.

The tensor field B_0 satisfies the following identities:

$$\begin{aligned} & B_{0,kjih} = B_{0,ihkj},\ B_{0,kjih} = -B_{0,jkih},\ B_{0,kjih} = -B_{0,kjhi}, \\ & B_{0,kjih} + B_{0,jikh} + B_{0,ikjh} = 0, \\ & B_{0,tjis}g^{ts} = \frac{2(n-2)}{n}R_{ji} - \frac{(n-2)r}{n^2}g_{ji}, \\ & B_{0,kjih}\varphi^{kh} = 0,\ B_{0,tsih}\varphi^{ts} = 0. \end{aligned}$$

Using these identities, (3.3) and (3.5), we can easily check that

$$\|B_0\|^2 = \|R\|^2 - \frac{8}{n^2}\|R_1\|^2 - \frac{2(n^2-2n-2)}{n^3(n+1)}r^2. \tag{3.6}$$

If the conformal curvature tensor field of M vanishes, that is, $B_0 = 0$, then from (3.3) and (3.5) we have

$$(n-2)R_{ji} = \frac{(n-2)r}{2n} g_{ji}.$$

Therefore we see that M is Einstein, provided $n \neq 2$.

4. $Spec^0 M$ and the geometry of M

Assume that M is a compact Kaehlerian manifold of dimension $2n$ and consider $Spec^0 M$. With the help of (3.4) and (3.6), the coefficient $a_{2,0}$ given by (2.3), may be written as follows :

$$(4.1) \quad a_{2,0} = \frac{1}{180} \int_M \left[\|B_0\|^2 + \frac{8-n^2}{n^2} \|Q\|^2 \right] dM + \frac{C_0(n)}{180} \int_M r^2 dM,$$

where $C_0(n)$ is constant depending only on n and $C_0(n) > 0$.

We shall often use the following Lemma 4.1.

LEMMA 4.1. ([7]) *Let (M, g) and (M', g') be compact orientable Riemannian manifolds with $Vol(M, g) = Vol(M', g')$ and $\int_M r\, dM = \int_{M'} r' dM'$. If r' =constant, then $\int_M r^2 dM \geq \int_{M'} r'^2 dM'$ with equality if and only if r = constant = r'.*

THEOREM 4.2. *Let M and M' be compact Kaehlerian manifolds. Assume that $Spec^0 M = Spec^0 M'$. Then $\dim M = \dim M' = 2n = m$, and*

(a) for $m = 2$, the conformal curvature tensor field of M vanishes and r = constant if and only if the conformal curvature tensor field of M' vanishes and r' = constant = r.

(b) when M and M' are Einstein and $m \geq 4$, the conformal curvature tensor field of M vanishes if and only if the conformal curvature tensor field of M' vanishes, $r = r'$.

Proof. Because of (2.1) and (2.2), $a_{0,0} = a'_{0,0}$ and $a_{1,0} = a'_{1,0}$ imply $Vol(M) = Vol(M')$ and $\int_M r\, dM = \int_{M'} r' dM'$. Moreover, by virtue of

(4.1), $a_{2,0} = a'_{2,0}$ yields

$$\begin{aligned}(4.2)\quad \int_M \left[\|B_0\|^2 + \frac{8-n^2}{n^2}\|Q\|^2\right] dM + C_0(n)\int_M r^2 dM \\ = \int_{M'} \left[\|B'_0\|^2 + \frac{8-n^2}{n^2}\|Q'\|^2\right] dM' + C_0(n)\int_{M'} r'^2 dM'.\end{aligned}$$

(a) If $n = 1$ and ${B_0}' = 0$, then $Q' = 0$ and it follows from (4.2) that

$$\int_M \left[\|B_o\|^2 + 7\|Q\|^2\right] dM + C_0(n)\Big(\int_M r^2 dM - \int_{M'} r'^2 dM'\Big) = 0.$$

which, by $r' =$ constant and Lemma 4.1, gives our assertion.

(b) Let $Q = 0$ and $Q' = 0$. Then r and r' are constant for $n \geq 2$. Thus by Lemma 4.1, we obtain $r = r'$ and it follows from (4.2) that

$$\int_M \|B_0\|^2 dM = \int_{M'} \|{B_0}'\|^2 dM'.$$

If the conformal curvature tensor field of M' vanishes, that is, ${B_0}' = 0$, then from the above equation, we obtain $B_0 = 0$. This completes the proof of our Theorem 4.2.

5. *Spec*^1M and the geometry of M

Assume that M is a compact Kaehlerian manifold of dimension $2n$ and consider *Spec*^1M. With the help of (3.4) and (3.6), the coefficient $a_{2,1}$ given by (2.6) reduces to

$$\begin{aligned}(5.1)\qquad a_{2,1} =& \frac{1}{180}\int_M \left[(2n-15)\|B_0\|^2 - A(n)\|Q\|^2\right] dM \\ &+ \frac{C_1(n)}{180}\int_M r^2 dM,\end{aligned}$$

where $A(n) = \frac{1}{n^2}(2n^3 - 90n^2 - 16n + 120)$ and $C_1(n) = \frac{1}{n(n+1)}(5n^3 - 26n^2 + 18n + 15)$.

THEOREM 5.1. *Let M and M' be compact Kaehlerian manifolds. Assume that $Spec^1 M = Spec^1 M'$. Then* $\dim M = \dim M' = 2n = m$, *and*

(a) *for* $16 \le m \le 90$, *the conformal curvature tensor field of M vanishes if and only if the conformal curvature tensor field of M' vanishes,* $r = r'$,

(b) *when M and M' are Einstein and $m \ge 4$, the conformal curvature tensor field of M vanishes if and only if the conformal curvature tensor field of M' vanishes,* $r = r'$.

Proof. Because of (2.4) and (2.5), $a_{0,1} = a'_{0,1}$ and $a_{1,1} = a'_{1,1}$ imply $Vol(M) = Vol(M')$ and $\int_M r dM = \int_{M'} r' dM'$. Moreover, by virtue of (5.1), $a_{2,1} = a'_{2,1}$ yields

$$(5.2)\quad \int_M \left[(2n-15)\|B_0\|^2 - A(n)\|Q\|^2\right] dM + C_1(n)\int_M r^2 dM$$
$$= \int_{M'} \left[(2n-15)\|B_0{}'\|^2 - A(n)\|Q'\|^2\right] dM' + C_1(n)\int_{M'} r'^2 dM'.$$

Using (5.2) and Lemma 4.1, we easily obtain our assertions.

THEOREM 5.2. *Let M and M' be compact Kaehlerian manifolds. Assume that $Spec^0 M = Spec^0 M'$ and $Spec^1 M = Spec^1 M'$. Then* $\dim M = \dim M' = 2n = m$, *and*

(a) *for $m \ge 4$, M is Einstien if and only if M' is Einstien,* $r' = r$,

(b) *for $m \ge 6$, the conformal curvature tensor field of M vanishes if and only if the conformal curvature tensor field of M' vanishes,* $r = r'$.

Proof. Because of (2.1) and (2.2), $a_{0,0} = a'_{0,0}$ and $a_{1,0} = a'_{1,0}$ imply $Vol(M) = Vol(M')$ and $\int_M r dM = \int_{M'} r' dM'$. Moreover, by virtue of (2.3) and (2.6), $a_{2,0} = a'_{2,0}$ and $a_{2,1} = a'_{2,1}$ yield

$$(5.3)\qquad \int_M \left[5\|R\|^2 + 13r^2\right] dM = \int_{M'} \left[5\|R'\|^2 + 13r'^2\right] dM',$$

$$(5.4)\qquad \int_M \left[10\|R_1\|^2 + r^2\right] dM = \int_{M'} \left[10\|R_1{}'\|^2 + r'^2\right] dM'.$$

(a) By (3.4), the equation (5.4) may be written as

$$\int_M \|Q\|^2 dM - \int_{M'} \|Q'\|^2 dM' + \frac{n+5}{10n}\Big(\int_M r^2 dM - \int_{M'} r'^2 dM'\Big) = 0.$$

If $Q' = 0$, then r' is constant for $n \geq 2$. Thus, by Lemma 4.1, the last equality leads to $Q = 0$ and r = constant = r'.

(b) Using (3.6), we write (5.3) in the form

$$\begin{aligned}&\int_M \Big[5\|B_0\|^2 dM + \frac{40}{n^2}\|R_1\|^2 + C_2(n) r^2\Big] dM \\ &= \int_{M'} \Big[5\|B_0{}'\|^2 + \frac{40}{n^2}\|R_1{}'\|^2 + C_2(n) r'^2\Big] dM',\end{aligned}$$

where $C_2(n) = \frac{1}{n^3(n+1)}(13n^4 + 13n^3 + 10n^2 - 20n - 20)$. This equality together with (5.4) gives

$$\begin{aligned}&\int_M \|B_0\|^2 dM - \int_{M'} \|B_0{}'\|^2 dM' \\ &+ \frac{13n^4 + 13n^3 + 6n^2 - 24n - 20}{5n^3(n+1)}\Big(\int_M r^2 dM - \int_{M'} r'^2 dM'\Big) = 0.\end{aligned}$$

Assume that $B_0{}' = 0$. Then r' is constant for $n \geq 3$. In view of the Lemma 4.1, the last equation yields now $B_0 = 0$ and r = constant = r'. This completes the proof of the Theorem.

6. *Spec*^2M and the geometry of M

Assume that M is a compact Kaehlerian manifold of dimension $2n$ and consider *Spec*^2M. With the help of (3.4) and (3.6) the coefficient $a_{2,2}$, given by (2.9), may be written as follows :

$$\begin{aligned} a_{2,2} =& \frac{1}{180}\int_M \Big[(n-8)(2n-15)\|B_0\|^2 \\ &- \frac{2n^4 - 181n^3 + 524n^2 + 248n - 960}{n^2}\|Q\|^2\Big] dM \\ &+ \frac{1}{180}\int_M \frac{10n^4 - 117n^3 + 362n^2 - 183n - 60}{2n(n+1)} r^2 dM. \end{aligned} \tag{6.1}$$

THEOREM 6.1. *Let M and M' be compact Kaehlerian manifolds. Assume that $Spec^2 M = Spec^2 M'$. Then* $\dim M = \dim M' = 2n = m$, *and*

(a) *for $m = 6, 8, 14$ or $18 \le m \le 174$, the conformal curvature tensor field of M vanishes if and only if the conformal curvature tensor field of M' vanishes, $r = r'$.*

(b) *for $m = 16$, M is Einstien if and only if M' is Einstien, $r' = r$,*

(c) *when M and M' are Einstein and $m \ge 4$ and $m \ne 16$, the conformal curvature tensor field of M vanishes if and only if the conformal curvature tensor field of M' vanishes, $r = r'$.*

Proof. The proof is based on the equalities $a_{0,2} = a'_{0,2}$, $a_{1,2} = a'_{1,2}$ and $a_{2,2} = a'_{2,2}$, where the coefficients are given by (2.7), (2.8) and (6.1). The idea of the proof is similar to that of Theorem 4.2. Therefore, we shall omit the details.

THEOREM 6.2. *Let M and M' be compact Kaehlerian manifolds. Assume that $Spec^0 M = Spec^0 M'$ and $Spec^2 M = Spec^2 M'$. Then* $\dim M = \dim M' = 2n = m$, *and*

(a) *for $m = 4$ or $m \ge 14$, M is Einstein if and only if M' is Einstien, $r' = r$,*

(b) *when for $m \ge 6$, the conformal curvature tensor field of M vanishes if and only if the conformal curvature tensor field of M' vanishes, $r = r'$.*

Proof. Because of (2.1) and (2.2), $a_{0,0} = a'_{0,0}$ and $a_{1,0} = a'_{1,0}$ imply $Vol(M) = Vol(M')$ and $\int_M r dM = \int_{M'} r' dM'$. Moreover, by virtue of (2.3) and (2.9), $a_{2,0} = a'_{2,0}$ and $a_{2,2} = a'_{2,2}$ yield

$$(6.2) \qquad \int_M [(5n-14)\|R\|^2 + (13n-40)r^2]dM = \int_{M'} [(5n-14)\|R'\|^2 + (13n-40)r'^2]dM',$$

$$(6.3) \qquad \int_M [2(5n-14)\|R_1\|^2 + (n-10)r^2]dM = \int_{M'} [2(5n-14)\|R_1'\|^2 + (n-10)r'^2]dM'.$$

(a) By (3.4), the equation (6.3) may be written as

$$\int_M 2(5n-14)\|Q\|^2 dM - \int_{M'} 2(5n-14)\|Q'\|^2 dM' + \frac{(n-7)(n+2)}{n}\Big(\int_M r^2 dM - \int_{M'} r'^2 dM'\Big) = 0.$$

Let $Q' = 0$, then r' is constant for $n \geq 2$. Thus, by Lemma 4.1, our last equality leads to $Q = 0$ and $r = \text{constant} = r'$ for $n = 2$ or $n \geq 7$.

(b) Using (3.6), we rewrite (6.2) in the form

$$\int_M \Big[(5n-14)\|B_0\|^2 + \frac{8(5n-14)}{n^2}\|R_1\|^2 + C_3(n)r^2\Big] dM = \int_{M'} \Big[(5n-14)\|B_0'\|^2 + \frac{8(5n-14)}{n^2}\|R_1'\|^2 + C_3(n)r'^2\Big] dM',$$

where $C_3(n) = \frac{1}{n^3(n+1)}(13n^5 - 27n^4 - 30n^3 - 48n^2 + 36n + 56)$. This equality together with (6.3) gives

$$\int_M (5n-14)\|B_0\|^2 dM - \int_{M'} (5n-14)\|B_0'\|^2 dM' + \frac{13n^5 - 27n^4 - 34n^3 - 12n^2 + 76n + 56}{n^3(n+1)}\Big(\int_M r^2 dM - \int_{M'} r'^2 dM'\Big) = 0.$$

If $B_0' = 0$, then r' is constant for $n \geq 3$. Thus, by Lemma 4.1, the last equation yields $B_0 = 0$ and $r = \text{constant} = r'$. This completes the proof of the Theorem.

References

1. M. Berger, P. Gauduchon et E. Mazet, *Le Spectre d' une Variété Riemannienne*, Lecture Notes in Mathematics 194, Springer-Verlag, 1971.
2. H. Kitahara, K. Matsuo and J. S. Pak, *A conformal curvature tensor field on Hermitian manifolds*, J. Korean Math. Soc. **27** (1990), 7-17.
3. H. Kitahara, K. Matsuo and J. S. Pak, *Appendium: A conformal curvature tensor field on Hermitian manifolds*, Bull. Korean Math. Soc. **27** (1990), 27-30.
4. J. S. Pak, J. C. Jeong and W.-T. Kim, *The contact conformal curvature tensor field and the spectrum of the Laplacian*, J. Korean Math. Soc. **28** (1991), 267-274.

5. J. S. Pak, J.-H. Kwon and K.-H. Cho, *On the spectrum of the Laplacian in cosymplectic manifolds*, Nihonkai Math. J. **3** (1992), 49-66.
6. V. K. Patodi, *Curvature and the fundamental solution of the heat operator*, J. Indian Math. Soc. **34** (1970), 269-285.
7. S. Tanno, *Eigenvalues of the Laplacian of Riemannian manifolds*, Tôhoku Math. J. **25** (1973), 391-403.
8. Gr. Tsagas, *On the spectrum of the Laplace operator for the exterior 2-forms*, Tensor N. S. **33** (1979), 94-96.
9. S. Yamaguchi and G. Chūman, *Eigenvalues of the Laplacian of Sasakian manifolds*, TRU Math. **15-2** (1979), 31-41.
10. K. Yano and M. Kon, *Structures on manifolds*, World Scientific, 1984.

JIN SUK PAK
DEPARTMENT OF MATHEMATICS, KYUNGPOOK NATIONAL UNIVERSITY, TAEGU 702-701, KOREA

KWAN-HO CHO AND JUNG-HWAN KWON
DEPARTMENT OF MATHEMATICS EDUCATION, TAEGU UNIVERSITY, TAEGU 705-714, KOREA