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MULTIPLICATION MODULES AND CHARACTERISTIC SUBMODULES

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1. Introduction

In this note all rings are commutative rings with identity and all modules are unital. Let R be a ring. An R-module M is called a *multiplication module* if for every submodule N of M there exists an ideal I of R such that $N = IM$. Clearly the ring R is a multiplication module as a module over itself. Also, it is well known that invertible and more generally projective ideals of R are multiplication R-modules (see [11, Theorem 1]).

If N is a submodule of M, then $(N : M)$ denotes the ideal Ann_R (M/N) of R, that is $(N : M) = \{r \in R : rM \subseteq N\}$. Let N be a submodule of a multiplication module M. Then there exists an ideal I of R such that $N = IM$. Note that $I \subseteq (N : M)$ and $N = IM \subseteq (N : M)M \subseteq N$ so that $N = (N : M)M$. It follows that an R-module M is a multiplication module if and only if $N = (N : M)M$ for all submodules N of M.

Let R be a ring and let M be an R-module. A submodule N of M will be called *characteristic submodule* if $\varphi(N) = N$ for all automorphisms φ of M. The R-module M will be called *finitely projective* if for every finitely generated submodule N of M, there exist a positive integer n, elements $m_i \in M$ $(1 \leq i \leq n)$ and R-homomorphisms $\theta_i : M \longrightarrow R$ $(1 \leq i \leq n)$ such that $x = \theta_1(x)m_1 + \cdots + \theta_n(x)m_n$, for all x in N.

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Note that any finitely generated finitely projective module is projective by the Dual Basis Lemma.

In this paper, we give some characterizations of finitely generated multiplication modules in terms of characteristic submodules and finitely projective R-modules (see Theorem 2 and Theorem 5). Moreover, we investigate further the relationship between multiplication modules and small submodules (see Theorem 9 and Theorem 10). Finally, we remark that we shall adopt the following notations : R is an arbitrary ring, M is an R-module, $A = Ann_R(M) = \{r \in R : rM = 0\}$ is the annihilator of M. $End({}_RM)$ and $Aut({}_RM)$ denote the endomorphism ring and automorphism group of an R-module M, respectively. In addition we set $M^+ = Hom_R(M, R/A)$.

2. Multiplication modules and characteristic submodules

Our starting point is the following easy lemma about characteristic submodule.

LEMMA 1. *Let M be a multiplication module. Then every submodule of M is a characteristic submodule.*

Proof. Let N be any submodule of M. Then $N = IM$ for some ideal I of R. Let $\varphi \in Aut({}_RM)$. Then $\varphi(N) = \varphi(IM) = I\varphi(M) = IM = N$.

THEOREM 2. *Let M be a finitely generated module. Then the following statements are equivalent.*

(i) *M is a multiplication module.* .

(ii) *Every submodule of every homomorphic image of M is a characteristic.*

Proof. (i) $\Longrightarrow$ (ii) : By Lemma 1, since every homomorphic image of M is a multiplication module.

(ii) $\Longrightarrow$ (i) : Let P be any maximal ideal of R. Then M/PM is a vector space over the field R/P, and hence M/PM is completely reducible. Since every submodule of M/PM is characteristic, it follows that M/PM is cyclic (see [4, Corollary 10]). By [2, Corollary 1.5] , M is a multiplication module.

The next two results are well known in [9].

LEMMA 3. *Suppose that M is a multiplication module. Then M is a finitely projective (R/A)-module.*

LEMMA 4. *Suppose that $Aut({}_RM)$ is commutative. Then*

$$\theta_1(m_1)\theta_2(m_2)m_3 = \theta_1(m_3)\theta_2(m_1)m_2$$

for all $m_i \in M, \theta_j \in M^+$ $(1 \leq i \leq 3, 1 \leq j \leq 2)$.

Let R be a ring and M an R-module. Then M is said to be *distributive* if $X \cap (Y + Z) = (X \cap Y) + (X \cap Z)$, for all submodules X, Y, Z of M. A ring R is said to be *arithmetical* if R considered as a module over itself is distributive.

THEOREM 5. *Let R be an arithmetical ring and M a finitely generated R-module. Then the following statements are equivalent.*

(i) *M is a distributive module.*
(ii) *M is a multiplication module.*
(iii) *M is a finitely projective (R/A)-module and every submodule of M is characteristic.*
(iv) *M is a finitely projective (R/A)-module and $Aut({}_RM)$ is commutative.*
(v) *M is a finitely projective (R/A)-module and $\theta(m)M \subseteq Rm$ for all $m \in M$ and every R-homomorphism $\theta : M \longrightarrow R/A$.*

Proof. (i) $\Longrightarrow$ (ii) : Since M is a finitely generated distributive module, $N = (N : M)M$ for any submodule N of M by [5, Lemma 3.2] and hence M is a multiplication module.

(ii) $\Longrightarrow$ (i) : By [6, Proposition 1.2].

(ii) $\Longrightarrow$ (iii) : By Lemma 1 and Lemma 3.

(iii) $\Longrightarrow$ (iv) : Clearly $Aut({}_RM)$ is a group under the operation of composition of functions. Let $f, g \in Aut({}_RM)$ and let $m \in M$. Because Rm is a characteristic submodule of M, there exists elements $r, s \in R$ such that $f(m) = rm$ and $g(m) = sm$. Thus

$$fg(m) = f(sm) = r(sm) = (rs)m = (sr)m = s(rm) = gf(m).$$

Hence $Aut({}_RM)$ is a commutative group.

(iv) $\Longrightarrow$ (v) : Let $m \in M$, $\theta \in M^+$. Let $y \in M$. Then

$$y = \theta_1(y)m_1 + \theta_2(y)m_2 + \cdots + \theta_n(y)m_n,$$

for some positive integer n, $m_i \in M, \theta_i \in M^+$ $(1 \le i \le n)$. By Lemma 4,

$$\begin{aligned}\theta(m)y &= \theta(m)\theta_1(y)m_1 + \theta(m)\theta_2(y)m_2 + \cdots + \theta(m)\theta_n(y)m_n \\ &= \theta(y)\theta_1(m_1)m + \theta(y)\theta_2(m_2)m + \cdots + \theta(y)\theta_n(m_n)m \in Rm.\end{aligned}$$

This implies $\theta(m)M \subseteq Rm$.

(v) $\Longrightarrow$ (ii) : Let $m \in M$. Since M is a finitely projective (R/A)-module, there exist a positive integer n, $m_i \in M$ and $\theta_i \in M^+(1 \le i \le n)$ such that

$$m = \theta_1(m)m_1 + \cdots + \theta_n(m)m_n.$$

By hypothesis, $\theta_i(m) \in (Rm : M)$ $(1 \le i \le n)$. Thus $m \in (Rm : M)M$ for all $m \in M$. By [2, Proposition 1.1], M is a multiplication module.

REMARK. Let R be a commutative ring with 1 and M be a unitary R-module. Observe that $End({}_RM)$ is not commutative in general, for example, if M is free of rank $n > 1$, then $End({}_RM)$ is isomorphic to the ring of $n \times n$ matrices with entries in R, and this ring is not commutative. (See also, [7] p.67. Exercise 3).

On the other hand, if M is a multiplication module, then $End({}_RM)$ is commutative, because every submodule of M is fully invariant (see, for example, [4, Proposition 7]).

3. Multiplication modules and small submodules

A *chain* of submodules of a module M is a sequence (M_i) $(1 \le i \le n)$ of submodules of M such that $M = M_0 \supset M_1 \supset \cdots \supset M_n = 0$ (strict inclusions). The *length* of the chain is n (the number of "links"). A *composition series* of M is a maximal chain, this is one in which no extra submodules can be inserted : this is equivalent to saying that each quotient M_{i-1}/M_i $(1 \le i \le n)$ is simple (that is, has no submodules except 0 and itself).

THEOREM 6. *Let M be a multiplication module and let R be a ring satisfying the descending chain condition on ideals containing $Ann_R(M)$, then M has a composition series.*

Proof. Let $A_1 \supseteq A_2 \supseteq \cdots$ be any descending chain on submodules of M. Then

$$(A_1 : M) \supseteq (A_2 : M) \supseteq \cdots .$$

Since $Ann_R(M) = (O : M) \subseteq (A_i : M)$ for all i, there exists an integer n such that $(A_n : M) = (A_i : M)$ for all $i \geq n$ by hypothesis. This implies that $(A_n : M)M = (A_i : M)M$ for all $i \geq n$. Since M is a multiplication module, $A_n = A_i$ for all $i \geq n$. Thus M is artinian. Since artinian multiplication module is cyclic [2, Corollary 2.9], M is cyclic and hence $R/(Ann_R(M)) \cong M$. Thus the ring $R/(Ann_R(M))$ is artinian. Since any artinian ring is noetherian, it is also noetherian so that M is noetherian. Thus M satisfies both chain conditions. By [1, p.77, Proposition 6.8], M has a composition series.

Theorem 6 has the following two corollaries and their proofs are immediate by the proof of the theorem.

COROLLARY 7. *Let M be a multiplication module and let R be a ring satisfying the hypothesis of Theorem 6, then M has a finite length.*

COROLLARY 8. *Let M be a multiplication module and let R be a ring satisfying the hypothesis of Theorem 6, then M is noetherian.*

A submodule K of a module M is called *small (or superflous)* in M provided for all submodules L of M, $K + L = M$ implies $L = M$. An ideal of a ring R is called *small* if it is a small submodule of R where considered as an R-module. On the other hand, if every proper submodule of an R-module M is small in M, then M is called a *hollow module.*

THEOREM 9. *Let M be a finitely generated faithful multiplication R-module. Then M is hollow if and only if every proper ideal of R is small.*

Proof. Suppose M is hollow. Let I be any proper ideal of R such that $I + J = R$ for all ideals J of R. Then

$$(I + J)M = IM + JM = M.$$

Since I is proper, $IM \neq M$ by [2, Theorem 3.1]. By hypothesis, $JM = M$ and hence $J = R$, again by [2, Theorem 3.1]. Therefore I is small.

Conversely, let $A(\neq M)$ and B be any submodule of M with $A+B = M$. Then $A = (A:M)M$ and $B = (B:M)M$. Thus

$$M = A + B = (A:M)M + (B:M)M = ((A:M) + (B:M))M.$$

This implies that $(A:M) + (B:M) = R$ by [2, Theorem 3.1]. If $(A:M) = R$, then $A = (A:M)M = RM = M$. This is a contradiction and so $(A:M) \neq R$. By hypothesis, $(B:M) = R$. This shows that $B = (B:M)M = RM = M$. Accordingly M is hollow.

THEOREM 10. *Let M be a finitely generated faithful multiplication R-module. A submodule N of M is small if and only if there exists a small ideal I of R such that $N = IM$.*

Proof. Suppose N is a small submodule of M. Then there exists an ideal A of R such that $N = AM$. Suppose $A + B = R$ for all ideals B of R. Then

$$N + BM = AM + BM = (A+B)M = RM = M.$$

By hypothesis, $BM = M$ and hence $B = R$ (see [2, Theorem 3.1]). This implies that A is a small ideal of R.

Conversely, suppose that I is a small ideal of R. Let C be a submodule of M such that $IM + C = M$. It is sufficient to show that $C = M$. Since M is a multiplication module, there exists an ideal K of R such that $C = KM$ and hence

$$(I+K)M = IM + KM = M.$$

By [2, Theorem 3.1], $I + K = R$ and hence $K = R$. Thus $C = M$.

We close this section with one more result about projective module.

THEOREM 11. *Let M be an R-module generated by the set $\{a_j | j \in I\}$ and let $S' = \{x_j | j \in I\}$. Let F be the free R-module generated by*

S'. If M is a projective R-module, then $F = Im\theta \oplus Ker\theta$ for some $\theta \in End({}_RM)$.

Proof. Consider the exact sequence

$$O \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow O$$

where $f : F \longrightarrow M$ is defined on the generators by $f(x_j) = a_j$ and $K = Kerf$. It is clear that K is an R-module which is a submodule of F. Since M is a projective module, there exists $\theta \in End({}_RF)$ such that $\theta^2 = \theta$ (see [8], p.18). First we show that $F = Im\theta + Ker\theta$. Clearly $Im\theta + Ker\theta \subseteq F$. Let $a \in F$, then $\theta(a) = b$ for some $b \in F$. This implies

$$\theta(a) = \theta^2(a) = \theta(b)$$

and hence $a \in Ker\theta + Im\theta$. It can easily be checked that $Im\theta \cap Ker\theta = 0$. This completes the proof.

COROLLARY 12. *Let M and F be R-modules satisfying the hypothesis of Theorem 11. If M is a projective R-module, then there exists $g \in End({}_RF)$ such that $fgf = f$ for some $f \in End({}_RF)$.*

Proof. By Theorem 11 and [12, Lemma 3.1].

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