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ON SPANNING COLUMN RANK OF MATRICES OVER SEMIRINGS

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A *semiring* is a binary system $(\mathbf{S}, +, \times)$ such that $(\mathbf{S}, +)$ is an Abelian monoid (*identity* 0), $(\mathbf{S}, x)$ is a monoid (*identity* 1), $\times$ distributes over $+, 0 \times s = s \times 0 = 0$ for all s in $\mathbf{S}$, and $1 \neq 0$. Usually $\mathbf{S}$ denotes the system and $\times$ is denoted by juxtaposition. If $(\mathbf{S}, \times)$ is Abelian, then $\mathbf{S}$ is commutative. Thus all rings are semirings. Some examples of semirings which occur in combinatorics are Boolean algebra of subsets of a finite set (with addition being union and multiplication being intersection) and the nonnegative integers (with usual arithmetic). The concepts of matrix theory are defined over a semiring as over a field. Recently a number of authors have studied various problems of semiring matrix theory. In particular, Minc [4] has written an encyclopedic work on nonnegative matrices.

In this note, we will have an equivalent definition of column rank of matrices over the nonnegative integers or binary Boolean algebra. We call it "spanning column rank", which enables us to calculate the column rank of matrices easily. We also point out that an example in [2] for column rank is wrong and we give a correct example. For our purpose, we will introduce some definitions and notations.

Let $\mathbf{S}$ denote a semiring and $M_{m,n}$ $(\mathbf{S})$ denote the set of $m \times n$ matrices with entries in $\mathbf{S}$. If V is a nonempty subset of $M_{m,1}$ $(\mathbf{S})$ that is closed under addition and multiplication by scalars, then V is called a *vector space* over $\mathbf{S}$. As with fields, a *basis* for a vector space V is a spanning subset of least cardinality. That cardinality is the *dimension*, $\dim(V)$, of V.

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We'll use the notation $< U >$ to denote the subspace spanned by the subset U of a vector space.

The ***column rank***, $c(A)$, of a nonzero $m \times n$ matrix A over $\mathbf{S}$ is the dimension of the vector space $< A >$ spanned by its columns.

The ***spanning column rank***, $sc(A)$, is the minimum number of the columns of A which span the vector space $< A >$. The zero matrix is assigned column rank and spanning column rank 0 respectively.

It follows that

$$0 \leq c(A) \leq sc(A) \leq n \tag{1}$$

for all $m \times n$ matrices A over $\mathbf{S}$.

The spanning column rank may actually exceed its column rank over some semirings. For example, let $\mathbf{S} = (\mathbf{Z}[\sqrt{7}])^+$, the semiring of nonnegative elements of $\mathbf{Z}[\sqrt{7}]$, where $\mathbf{Z}$ is the ring of integers. Consider a 1×2 matrix $A = [3-\sqrt{7}, \sqrt{7}-2]$ over $\mathbf{S}$. Since $(3-\sqrt{7}) + (\sqrt{7}-2) = 1, \{1\}$ is a spanning set of the column space of A. So $c(A) = 1$. But $sc(A) = 2$ since $3-\sqrt{7} \neq \alpha(\sqrt{7}-2)$ and $\sqrt{7}-2 \neq \alpha(3-\sqrt{7})$ for any α in $\mathbf{S}$.

For each $x \in \mathbf{S}$, let $\bar{x}$, its *pattern*, be 1 if $x \neq 0$ and 0 otherwise. If $A \in M_{m,n}(\mathbf{S})$, define $\bar{A}$, the *pattern* of A, to be $[\,\bar{a}_{ij}]$, the $m \times n$ matrix of patterns of the entries of A.

A set U of vectors over $\mathbf{S}$ is *linearly dependent* if for some $\mathbf{a} \in U$, $\mathbf{a}$ can be spanned by the set $U - \{\mathbf{a}\}$. Otherwise U is *linearly independent.*

Thus, for a nonzero matrix $A \in M_{m,n}(\mathbf{S})$, $sc(A)$ is the minimum number of linearly independent columns of A which generate the vector space $< A >$. Hence we have:

LEMMA 1. *If the columns of $A \in M_{m,n}(\mathbf{S})$ are linearly independent, then $sc(A)=n$.*

In the followings, $\mathbf{Z}^+$ denotes the semiring of nonnegative integers and $\mathbf{B}$ the binary Boolean algebra. For $a, b \in \mathbf{S}$, if $a = b + x$ for some $x \in \mathbf{S}$ we write $a \geq b$. The relation $\geq$ is extendable entrywise to vectors and matrices.

LEMMA 2. *Let a semiring $\mathbf{S}$ be $\mathbf{Z}^+$ or $\mathbf{B}$. Suppose U and V are nonempty sets of vectors in $\mathbf{S}^m (= M_{m,1}(\mathbf{S}))$, and U is linearly independent. Then $< U > = < V >$ implies that for all $\mathbf{a} \in U$, there exists*

$\mathbf{b} \in V$ and there exist nonzero scalars α and β such that $\mathbf{a} \geq \beta b$ and $\mathbf{b} \geq \alpha \mathbf{a}$.

Proof. Since $\mathbf{S}^m$ is finite dimensional and U is linearly independent, we may write $U = \{\mathbf{a}_1, \ldots, \mathbf{a}_p\}$ and $V = \{\mathbf{b}_1 \ldots, \mathbf{b}_q\}$ with no $\mathbf{a}_i = \mathbf{0}$. Let $k \leq$p. Since $< U > = < V >$, we have

$$\mathbf{a}_k = \sum_{i=1}^{q} \beta_i \mathbf{b}_i \text{ and } \mathbf{b}_i = \sum_{j=1}^{p} \alpha_{ij} \mathbf{a}_j$$

for some scalars β_i and α_{ij} in $\mathbf{S}$. Since

$$\mathbf{a}_k = \sum_{i=1}^{q} \beta_i (\sum_{j=1}^{p} \alpha_{ij} \mathbf{a}_j) = (\sum_{i=1}^{q} \beta_i \alpha_{ik}) \mathbf{a}_k + \sum_{j \neq k} (\sum_{i=1}^{q} \beta_i \alpha_{ij}) \mathbf{a}_j$$

and U is linearly independent, we have $\sum \beta_i \alpha_{ik} \neq 0$. So for some h, we have $\beta_h \alpha_{hk} \neq 0$. Thus $\mathbf{a}_k \geq \beta_h \mathbf{b}_h$ and $\mathbf{b}_h \geq \alpha_{hk} \mathbf{a}_k$.

THEOREM 3. *Let a semiring* $\mathbf{S}$ *be* $\mathbf{Z}^+$ *or* $\mathbf{B}$. *Then the column rank and the spanning column rank of an arbitrary matrix* $A \in M_{m,n}(\mathbf{S})$ *are the same.*

Proof. Assume that $c(A) = k$ and $sc(A)$=r. Let $U = \{ \mathbf{a}_{i(1)}, \ldots, \mathbf{a}_{i(r)}\}$ be a minimum set of linearly independent columns of A which spans the vector space $< A >$. Let $V = \{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ be a basis for the vector space $< A >$. That is, the elements of V are spanned by the columns of A. Then $< U > = < V >$. Thus Lemma 2 implies that for all $\mathbf{a}_{i(j)} \in U$, there exists $\mathbf{x}_h \in V$ and there exist nonzero scalars α and β such that $\mathbf{a}_{i(j)} \geq \beta \mathbf{x}_h$ and $\mathbf{x}_h \geq \alpha \mathbf{a}_{i(j)}$. That is, $\mathbf{a}_{i(j)} \geq \beta\alpha \mathbf{a}_{i(j)}$ and hence $\beta\alpha \leq 1$. Then $\alpha = \beta = 1$ and hence $\mathbf{a}_{i(j)} = \mathbf{x}_h$ for some $\mathbf{x}_h \in V$. Thus the spanning column rank r is not greater than the column rank k. By (1), we have r=k.

In [2], Beasley and Pullman gave a wrong example as follows;

EXAMPLE 4. ([2], Example 3.2.1) Let $\mathbf{S} = \mathbf{Z}^+$, and $A = [2, 3, 5, 7, 11, \ldots, p_n]$ where p_n is the nth prime integer in $\mathbf{S}$. Then $c(A) =$ n. But $c(\bar{A}) = 1$, where $\bar{A}$ is the pattern matrix of A.

Now we assert that the column rank of A in Example 4 is not n but

$$(2) \qquad c(A) = 2.$$

For, let p be an arbitrary prime integer larger than 3. Then $p-3$ is an even integer in $\mathbf{S}$. So $p-3$ is spanned by 2 over $\mathbf{S}$. Therefore p can be spanned by both 2 and 3 over $\mathbf{S}$. Since $\{2,3\}$ is the minimum set of linearly independent columns of A which spans the vector space $< A >$, we have $sc(A) = 2$. Hence $c(A) = 2$ by Theorem 3.

Now we give a correct example as follows;

EXAMPLE 5. Let $\mathbf{S} = \mathbf{Z}^+$ and n be any integer in $\mathbf{S}$. Consider

$$B = [n+1, n+2, \ldots, 2n].$$

Then the set of all the columns of B is linearly independent over $\mathbf{S}$ and is the minimum set which spans the vector space $< B >$. Thus $sc(B) = n$ and hence $c(B) = n$ by Theorem 3. But it is trivial that $c(\bar{B}) = 1$, where $\bar{B}$ is the pattern matrix of B.

PROPOSITION 6. *For A in $M_{m,n}(\mathbf{Z}^+)$, the spanning column rank of A is no less than that of its pattern matrix $\bar{A}$.*

Proof. Suppose $sc(A)$=k. Then there exist k linearly independent columns $\mathbf{a}_{i(1)}, \ldots, \mathbf{a}_{i(k)}$ of A which span all the columns of A. Since $\mathbf{Z}^+$ has no zero divisors and no negative elements, we have $\overline{\alpha + \beta} = \bar{\alpha} + \bar{\beta}$ and $\overline{\alpha\beta} = \bar{\alpha}\bar{\beta}$ for any $\alpha, \beta \in \mathbf{Z}^+$. For an arbitrary column $\mathbf{a}_j$ of A, we have $\mathbf{a}_j = \sum_{h=1}^{k} \beta_h \mathbf{a}_{i(h)}$ for some $\beta_h \in \mathbf{Z}^+$. Then $\bar{\mathbf{a}}_j = \sum_{h=1}^{k} \bar{\beta}_h \bar{\alpha}_{i(h)}$. Thus the set $\{\bar{\mathbf{a}}_{i(h)} \mid i = 1, \ldots, k\}$ of columns of $\bar{A}$ can span all the columns of $\bar{A}$. That is, $sc(\bar{A}) \leq k$. It implies that $sc(A) \geq sc(\bar{A})$.

The Example 5 shows that the column rank of a matrix can be greater strictly than that of its pattern matrix. But it is not the case for the matrices over general semirings. Here we give an interesting example.

EXAMPLE 7. For $n \geq 2$, let $\mathbf{B}_n$ be the finite nonbinary Boolean algebra of subsets of an n-element set $S_n = \{a_1, a_2, \ldots, a_n\}$. Union is denoted by $+$, and intersection by juxtaposition. 0 denotes the null set and 1 the set S_n. Let $\sigma_1 = \{a_1\}, \sigma_2 = \{a_2\}, \ldots, \sigma_n = \{a_n\}$ be the singleton subsets of S_n. Define $p_i = \sigma_1 + \sigma_2 + \ldots + \sigma_i$ and $q_j = \sigma_n + \sigma_{n-1} + \ldots + \sigma_{n-(j-1)}$ for $1 \le i, j \le n-1$. Then all σ_i, p_i and q_j are elements of $\mathbf{B}_n$. Consider an upper triangular matrix

$$D = \begin{bmatrix} q_1 & q_2 & q_3 & \cdots & q_{n-1} & 1 \\ 0 & q_2 p_{n-1} & q_3 p_{n-1} & \cdots & q_{n-1} p_{n-1} & p_{n-1} \\ 0 & 0 & q_3 p_{n-2} & \cdots & q_{n-1} p_{n-2} & p_{n-2} \\ \vdots & \vdots & 0 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & & q_{n-1} p_2 & p_2 \\ 0 & 0 & 0 & \cdots & 0 & p_1 \end{bmatrix}$$

over $\mathbf{B}_n$. Then each $k \times k$ principal submatrix P_k of D has spanning column rank 1 since the column that contains most nonzero entries among the columns of P_k can span all the columns of it over $\mathbf{B}_n$. But its pattern matrix $\bar{P}_k$ has spanning column rank k since each columns of it are linearly independent over binary Boolean algebra. Then $c(\bar{P}_k) = k$ by Theorem 3.

Now, consider another matrix E=diag$(\sigma_1, \sigma_2, \ldots, \sigma_n)$ over $\mathbf{B}_n$.

Then $sc(E) = n$ since all the columns of E are linearly independent over $\mathbf{B}_n$. But $c(E) = 1$ since a vector $\mathbf{x}$ which is the sum of all columns of E constitutes a spanning set of the vector space $< E >$ over $\mathbf{B}_n$. This shows that the spanning column rank of a matrix over $\mathbf{B}_n$ may be different to the column rank of it. Moreover the pattern matrix $\bar{E}$ of E is the identity matrix of order n, so we have $sc(\bar{E}) = n = c(\bar{E})$ by Lemma 1 and Theorem 3.

Thus the column rank or spanning column rank of the pattern matrix of a matrix may or may not be greater than those of the given matrix.

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