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# SOME EXISTENCE THEOREMS FOR GENERALIZED VECTOR VARIATIONAL INEQUALITIES

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## 1. Introduction and Preliminaries

Let $X$ and $Y$ be two normed spaces and $D$ a nonempty convex subset of $X$. Let $T : X \to L(X, Y)$ be a mapping, where $L(X, Y)$ is the space of all continuous linear mappings from $X$ into $Y$. And let $C : D \to 2^Y$ be a set-valued map such that for each $x \in D$, $C(x)$ is a convex cone in $Y$ such that $Int\ C(x) \neq \emptyset$ and $C(x) \neq Y$, where $Int$ denotes the interior.

In this paper we consider the following generalized vector variational inequality (GVVI):

(GVVI) Find $x_0 \in D$ such that $\langle T(x_0), x - x_0\rangle \notin -Int\ C(x_0)$ for all $x \in D$.

When for every $x \in D$, $C(x) = C$, where $C$ is a convex cone in $Y$ with $Int\ C \neq \emptyset$ and $C \neq Y$, then (GVVI) reduces to the following vector variational inequality (VVI) :

(VVI) Find $x_0 \in D$ such that $\langle T(x_0), x - x_0\rangle \notin -Int\ C$ for all $x \in D$.

(GVVI) was investigated by Chen [1], under the monotonicity and hemicontinuity of $T$. Many authors [2,3,4,10] have studied (VVI) in several directions. In particular, Yang [10] obtained the existence theorems

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for (VVI), by using the KKM-Fan theorem [5], under the continuity of $T$ and certain coercivity condition.

In this paper we establish some existence theorems for (GVVI), by using the particular form of generalized KKM theorems due to Park [7-9], under the continuity of $T$ and some coercivity condition more generalized than that of Yang [10].

Now we give some definitions needed in the sequel.

DEFINITION 1.1. Let $D$ be a subset of a topological vector space $X$. Then a set-valued map $G : D \to 2^X$ is called *KKM* if for each nonempty finite subset $\{x_1, \cdots, x_n\}$ of $D$, *co* $\{x_1, \cdots, x_n\} \subset \bigcup_{i=1}^n G(x_i)$, where *co* $\{x_1, \cdots, x_n\}$ is the smallest convex set generated by $\{x_1, \cdots, x_n\}$.

DEFINITION 1.2. Let $X$ and $Y$ be topological vector spaces and $G : X \to 2^Y$ a set-valued map. Then $G$ is said to be closed if the graph of $G : \{(x, y) : y \in G(x)\}$ is closed.

A ***convex space*** $X$ is a nonempty convex set (in a vector space) with any topology that induces the Euclidean topology on the convex hulls of its finite subsets. Thus, a convex subset $D$ of a topological vector space $X$ with the relative topology is automatically a convex space. For details of the convex space, see Lassonde [6].

We say that a subset $A$ of a topological space $X$ is *compactly closed* in $X$ if for every compact subset $K$ of $X$, the set $A \cap K$ is closed in $K$.

We need the following particular form of the generalized KKM theorems due to Park [7-9], which will be used in the proof of our Theorem 2.1 in Section 2.

THEOREM 0. *Let $X$ be a convex space, $K$ a nonempty compact subset of $X$ and $G : X \to 2^X$ a KKM map. Suppose that the following conditions are satisfied :*

(1) *for each $y \in X$, $G(y)$ is compactly closed ; and*

(2) *for each nonempty finite subset $N$ of $X$, there exists a nonempty compact convex subset $L_N$ of $X$ such that $N \subset L_N$ and $L_N \cap \bigcap\{G(y) : y \in L_N\} \subset K$.*

*Then we have*

$$K \cap \bigcap\{G(y) : y \in X\} \neq \emptyset.$$

## 2. Existence Theorems

Now, we give some existence theorems for (GVVI).

THEOREM 2.1. *Let $X$ and $Y$ be Banach spaces, $D$ a nonempty closed convex subset of $X$ and $K$ a nonempty compact subset of $X$ such that $K \cap D \neq \emptyset$. Let $C : D \to 2^Y$ a set-valued map such that for each $x \in D$, $C(x)$ is a closed convex cone in $Y$ with $Int\ C(x) \neq \emptyset$ and $C(x) \neq Y$. Let $W : D \to 2^Y$ be a closed set-valued map defined by for any $x \in D$, $W(x) = Y \backslash (-Int\ C(x))$ and $T : X \to L(X, Y)$ a mapping. Suppose that the following conditions are satisfied :*

*(1) $T$ is continuous ; and*

*(2) for each nonempty finite subset $N$ of $D$, there exists a nonempty compact convex subset $L_N$ of $D$ such that $N \subset L_N$ and for each $x \in L_N \backslash K$, there exists a $y \in L_N$ such that $\langle T(x), y - x\rangle \in -Int\ C(x)$.*

*Then (GVVI) is solvable. Furthermore, the solution set of (GVVI) is a compact subset of $K$.*

*Proof.* Define a set-valued map $F : D \to 2^D$ by for any $y \in D$,

$$F(y) = \{x \in D : \langle T(x), y - x\rangle \notin -Int\ C(x)\}.$$

Then $F$ is a KKM map on $D$. In fact, suppose that $N = \{x_1, \cdots, x_n\} \subset D$, $\sum_{i=1}^n \alpha_i = 1, \alpha_i \geq 0, i = 1, \cdots, n$ and $x = \sum_{i=1}^n \alpha_i x_i \notin \bigcup_{i=1}^n F(x_i)$. Then, we have $\langle T(x), x_i - x\rangle \in -Int\ C(x), i = 1, \cdots, n$. Thus we have

$$\begin{aligned}\langle T(x), x\rangle &= \langle T(x), \sum_{i=1}^n \alpha_i x_i\rangle \\ &= \sum_{i=1}^n \alpha_i \langle T(x), x_i\rangle \\ &\in \sum_{i=1}^n \alpha_i \langle T(x), x\rangle - Int\ C(x) \\ &= \langle T(x), x\rangle - Int\ C(x).\end{aligned}$$

Hence $0 \in\ Int\ C(x)$, which contradicts the assumption $C(x) \neq Y$. Therefore, $F$ is a KKM map on $D$.

We claim that $F$ is closed-valued. Indeed, let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $F(y)$ which converges to $x \in D$ for any fixed $y \in D$ . Then we have

$$\langle T(x_n), y - x_n\rangle \in W(x_n) \quad \text{for all } n. \tag{2.1}$$

Since $T$ is continuous, $T(x_n)$ converges to $T(x)$. Moreover we have

$$\begin{aligned}
&\|\langle T(x_n), y - x_n\rangle - \langle T(x), y - x\rangle\| \\
\le&\|\langle T(x_n), y - x_n\rangle - \langle T(x), y - x_n\rangle\| \\
&+ \|\langle T(x), y - x_n\rangle - \langle T(x), y - x\rangle\| \\
\le&\|T(x_n) - T(x)\| \cdot \|y - x_n\| + \|T(x)\| \cdot \|x_n - x\| \to 0 \quad (\text{ as } n \to \infty).
\end{aligned}$$

Hence $\langle T(x_n), y - x_n\rangle$ converges to $\langle T(x), y - x\rangle$. By (2.1) and the closedness of $W$, $\langle T(x), y-x\rangle \in W(x)$, i.e., $\langle T(x), y-x\rangle \notin -Int\ C(x)$. Thus $x \in F(y)$ and hence $F(y)$ is closed.

Further, note that assumption (2) implies that for each nonempty finite subset $N$ of $D$, there exists a nonempty compact convex subset $L_N$ of $D$ such that $N \subset L_N$ and for each $x \in L_N \backslash K$, there exists a $y \in L_N$ such that $x \notin F(y)$. Hence $L_N \cap \bigcap\{F(y) : y \in L_N\} \subset K$. Thus, the condition (2) of Theorem **0** holds.

Therefore, by Theorem **0**, $K \cap \bigcap\{F(y) : y \in D\} \neq \emptyset$. Thus there exists an $x_0 \in D$ such that

$$\langle T(x_0), x - x_0\rangle \notin -Int\ C(x_0) \text{ for any } x \in D.$$

Furthermore, by the condition (2), $\bigcap\{F(y) : y \in D\} \subset K$ and hence the solution set of (GVVI) is a compact subset of $K$.

COROLLARY 2.1. *In Theorem 2.1, the coercivity condition* (2) *can be replaced by the following without affecting its conclusion :*

(2)′ *there exist a nonempty compact subset $K$ of $X$ and a $y_0 \in K \cap D$ such that $\langle T(x), y_0 - x\rangle \in -Int\ C(x)$ for any $x \in D\backslash K$.*

*Proof.* It sufficies to show that (2)′ implies (2). In fact, for each nonempty finite subset $N$ of $D$, we let $L_N = co(N \cup (K \cap D)) \subset D$. By (2)′, for any $x \in L_N\backslash K \subset D\backslash K$, there exists a $y_0 \in K \cap D \subset L_N$ such that $\langle T(x), y_0 - x\rangle \in -Int\ C(x)$. Hence (2) holds.

From Corollary 2.1, we can obtain the following corollary (Theorem 1 in [10]);

COROLLARY 2.2. [10] *Let $X$ and $Y$ be Banach spaces, $D$ a nonempty closed convex subset of $X$, $C$ a closed convex cone in $Y$ with $Int\ C \neq \emptyset$ and $C \neq Y$ and $T : X \to L(X, Y)$ a mapping. Suppose that the following conditions are satisfied :*

(1) *$T$ is continuous ; and*

(2)″ *there exist a nonempty compact subset $K$ of $X$ and $y_0 \in K \cap D$ such that $\langle T(x), y_0 - x\rangle \in -Int\ C$ for any $x \in D \backslash K$.*

Then (VVI) is solvable. Furthermore, the solution set of (VVI) is a compact subset of $K$.

For $D = K$, Theorem 2.1 reduces to the following corollary ;

COROLLARY 2.3. *Let $X$ and $Y$ be Banach spaces, $D$ a nonempty compact convex subset of $X$, and $C : D \to 2^Y$ a set-valued map such that for each $x \in D$, $C(x)$ is a closed convex cone in $Y$ with $Int\ C(x) \neq \emptyset$ and $C(x) \neq Y$. Let $W : D \to 2^Y$ be a closed set-valued map defined by for any $x \in D$, $W(x) = Y \backslash(-Int\ C(x))$, and $T : X \to L(X, Y)$ a mapping. If $T$ is continuous, then (GVVI) is solvable. Furthemore, the solution set of (GVVI) is a compact subset of $D$.*

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