EPIMORPHISMS OF ANNIHILATORS
OF POOR M-COSEQUENCES

SANG-CHO CHUNG

0. Introduction

Let $R$ be a commutative ring with identity and $M$ an $R$-module. In ([Mt], 8), Matlis proved that, for a given $M$-sequence $\{x_1, \ldots, x_n\}$, the following map

$$M/(x_1^t, \ldots, x_n^t)M \rightarrow M/(x_1^{t+1}, \ldots, x_n^{t+1})M$$

is a monomorphism for all $t > 0$, and if $\{x_1, \ldots, x_n\}$ is an $M$-cosequence, then

$$\text{Ann}_M(x_1^{t+1}, \ldots, x_n^{t+1})R \rightarrow \text{Ann}_M(x_1^t, \ldots, x_n^t)R$$

is an epimorphism for all $t > 0$.

As a generalization of the first result of Matlis, in ([O], 3.2), O’carroll described that, when $\{y_1, \ldots, y_n\}$ is a poor $M$-sequence and $\{x_1, \ldots, x_n\}$ is a sequence of elements of $R$ such that $H[x_1 \ldots x_n]^T = [y_1 \ldots y_n]^T$ for some $n \times n$ lower triangular matrix $H$, the map

$$M/(x_1, \ldots, x_n)M \rightarrow M/(y_1, \ldots, y_n)M$$

is a monomorhism and $\{x_1, \ldots, x_n\}$ is also a poor $M$-sequence.

So we consider the dual case of O’carroll. That is, let $\{y_1, \ldots, y_n\}$ be a poor $M$-cosequence and $\{x_1, \ldots, x_n\}$ is a sequence of elements of $R$ such that $H[x_1 \ldots x_n]^T = [y_1 \ldots y_n]^T$ for some $n \times n$ lower triangular matrix $H$. We give an epimorphism

$$\text{Ann}_M(y_1, \ldots, y_n)R \rightarrow \text{Ann}_M(x_1, \ldots, x_n)R$$

and $\{x_1, \ldots, x_n\}$ is also a poor $M$-cosequence.

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1. Preliminaries

Throughout this note, \( R \) is a commutative ring with identity and \( M \) an \( R \)-module. We use \( T \) to denote matrix transpose and \( D_n(R) \) \((n \geq 1)\) to denote the set of \( n \times n \) lower triangular matrices over \( R \). For \( H \in D_n(R) \), \(|H|\) denotes the determinant of \( H \). Let \((a_1, \ldots, a_i)R\) be the ideal of \( R \) generated by \( \{a_1, \ldots, a_i\} \) and \((a_1, \ldots, a_i)M\) the submodule of \( M \) generated by \( \{ajm : j = 1, \ldots, i \text{ and } m \in M\} \).

Let \( \{x_1, \ldots, x_n\} \) be a sequence of elements of \( R \) and \( M \) an \( R \)-module. Then \( \{x_1, \ldots, x_n\} \) is said to be a poor \( M \)-sequence if multiplication by \( x_i \) on \( M/(x_1, \ldots, x_{i-1})M \) is a monomorphism for all \( i = 1, \ldots, n \) (where \( x_0 = 0 \)). If, in addition, \( M/(x_1, \ldots, x_n)M \neq 0 \), we call \( \{x_1, \ldots, x_n\} \) an \( M \)-sequence.

If \( b \) is an ideal of \( R \), we define \( \text{Ann}_M b = \{m \in M : bm = 0\} \). We have a dual definition; \( \{x_1, \ldots, x_n\} \) is said to be a poor \( M \)-cosequence if multiplication by \( x_i \) on \( \text{Ann}_M(x_1, \ldots, x_{i-1})R \) is an epimorphism for all \( i = 1, \ldots, n \) (where \( x_0 = 0 \)). Similarly, if \( \text{Ann}_M(x_1, \ldots, x_n)R \neq 0 \), \( \{x_1, \ldots, x_n\} \) is called an \( M \)-cosequence.

Let \( E \) be an injective envelope of the direct sum of all of the simple \( R \)-modules, and define the functor \( * \) by \( * = \text{Hom}(\cdot, E) \), then \( * \) is a faithfully exact contravariant functor; that is, a sequence of \( R \)-modules is exact if and only if its \( \cdot * \) is exact.

**Lemma 1.1.** Let \( R \) be a ring and \( M \) an \( R \)-module. Assume that \( N \) is an injective \( R \)-module and \( a \) a finitely generated ideal of \( R \). Then we have the following.

1. \( R/a \otimes_R \text{Hom}(M, N) \cong \text{Hom}(\text{Hom}(R/a, M), N) \).

2. If, in addition, \( a \) is generated by a poor \( M \)-cosequence and \( N = E \), then we have
   \[ a \otimes_R \text{Hom}(M, E) \cong \text{Hom}(\text{Hom}(a, M), E) \cong aM^* \).

3. If, in addition, \( a \) is generated by a poor \( M \)-sequence, then we have
   \[ a \otimes M \cong aM. \]

In particular, we have \( \text{Hom}(a \otimes M, E) \cong \text{Hom}(a, \text{Hom}(M, E)) \cong (aM)^* \).
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**Proof.** (1) Consider the following exact sequence

$$0 \longrightarrow a \longrightarrow R \longrightarrow R/a \longrightarrow 0.$$  

Then by ([Mm], p.14 Exercise 2.5(b)) $R/a$ is of finite presentation. Hence the assertion follows from ([R], 3.60).

(2) Assume that $a$ is generated by an $M$-cosequence. Note that the generators of $a$ forms an $M^*$-sequence by ([Mt], 5(2)), or the following Lemma 1.2(2). From the above short exact sequence, we have the following short exact sequence;

$$0 \longrightarrow \text{Hom}(R/a, M) \longrightarrow \text{Hom}(R, M) \longrightarrow \text{Hom}(a, M) \longrightarrow 0,$$

since $\text{Ext}_R^1(R/a, M) = 0$ by ([Mt], 4 and [BH], 1.1.12).

Hence we get the following exact sequence;

$$0 \longrightarrow \text{Hom}(\text{Hom}(a, M), E) \longrightarrow \text{Hom}(M, E) \longrightarrow \text{Hom}(\text{Hom}(R/a, M), E) \longrightarrow 0,$$

since $E$ is an injective $R$-module.

Now, from (1) and the short exact sequence in the proof of (1), we have the following commutative diagram;

$$
\begin{array}{ccc}
0 & \longrightarrow & a \otimes \text{Hom}(M, E) \\
& \downarrow & \downarrow \\
& R \otimes \text{Hom}(M, E) & \longrightarrow R/a \otimes \text{Hom}(M, E) & \longrightarrow 0 \\
& \text{Hom}(\text{Hom}(a, M), E) & \longrightarrow \text{Hom}(M, E) & \longrightarrow \text{Hom}(\text{Hom}(R/a, M), E) & \longrightarrow 0,
\end{array}
$$

since $\text{Tor}_1^R(R/a, \text{Hom}(M, E)) = 0$ by ([BH], 1.1.12). Hence the five lemma gives the first isomorphism.

Next since $R \otimes \text{Hom}(M, E) \cong M^*$ and $R/a \otimes \text{Hom}(M, E) \cong M^*/aM^*$, we have $a \otimes \text{Hom}(M, E) \cong aM^*$ from the top exact sequence of the above commutative diagram.

(3) Using the short exact sequence in the proof of (1) again, we have the following short exact sequence;

$$0 \longrightarrow a \otimes M \longrightarrow M \longrightarrow M/aM \longrightarrow 0,$$

since $\text{Tor}_1^R(R/a, M) = 0$ by ([BH], 1.1.12). Hence we obtain $a \otimes M \cong aM$. 

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Lemma 1.2. (cf. [Mt], 1, 5 and 6) (1) \( \{x_1, \ldots, x_n\} \) is a poor \( M \)-sequence if and only if \( \{x_1, \ldots, x_n\} \) is a poor \( M^* \)-cosequence.

(2) \( \{x_1, \ldots, x_n\} \) is a poor \( M \)-cosequence if and only if \( \{x_1, \ldots, x_n\} \) is a poor \( M^* \)-sequence.

(3) \( \{x_1, \ldots, x_n\} \) is a poor \( M \)-sequence if and only if \( \{x_1^{\alpha_1}, \ldots, x_n^{\alpha_n}\} \) is a poor \( M \)-sequence for any positive integers \( \alpha_1, \ldots, \alpha_n \).

(4) \( \{x_1, \ldots, x_n\} \) is a poor \( M \)-cosequence if and only if \( \{x_1^{\alpha_1}, \ldots, x_n^{\alpha_n}\} \) is a poor \( M \)-cosequence for any positive integers \( \alpha_1, \ldots, \alpha_n \).

Proof. ((1) and (2)) Note that for \( i = 1, \ldots, n \)

\[
(M/(x_1, \ldots, x_i)M)^* \cong \text{Hom}(R/(x_1, \ldots, x_i)R \otimes M, E) \\
\cong \text{Hom}(R/(x_1, \ldots, x_i)R, \text{Hom}(M, E)) \cong \text{Ann}_{M^*}(x_1, \ldots, x_i)R
\]

and

\[
(\text{Ann}_M(x_1, \ldots, x_i)R)^* \cong \text{Hom}(\text{Hom}(R/(x_1, \ldots, x_i)R, M), E) \\
\cong R/(x_1, \ldots, x_i)R \otimes \text{Hom}(M, E) \cong M^*/(x_1, \ldots, x_i)M^*
\]

by Lemma 1.1(1).

Then the results follow easily from the above isomorphisms, since \( \ast \) is faithfully exact.

(3) This follows from ([K], p.102 Exercise 12).

(4) The proof follows immediately from (2) and (3).

2. Main results

Lemma 2.1. ([O], 3.2) Let \( R \) be a ring and \( M \) an \( R \)-module. Consider two sequences \( \{x_1, \ldots, x_n\} \) and \( \{y_1, \ldots, y_n\} \) of elements of \( R \) such that

(i) \( H[x_1 \ldots x_n]^T = [y_1 \ldots y_n]^T \) for some \( H \in D_n(R) \), and

(ii) \( \{y_1, \ldots, y_n\} \) is a poor \( M \)-sequence.

Then the map from \( M/(x_1, \ldots, x_n)M \) to \( M/(y_1, \ldots, y_n)M \) induced by multiplication by \( |H| \) is a monomorphism and \( \{x_1, \ldots, x_n\} \) is also a poor \( M \)-sequence.
Theorem 2.2. Let $R$ be a ring and $M \in R$-module. Consider two sequences $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_n\}$ of elements of $R$ such that
(i) $H[x_1 \ldots x_n]^T = [y_1 \ldots y_n]^T$ for some $H \in D_n(R)$, and
(ii) $\{y_1, \ldots, y_n\}$ is a poor $M$-cosequence.

Then the map from $\text{Ann}_M(y_1, \ldots, y_n)R$ to $\text{Ann}_M(x_1, \ldots, x_n)R$ induced by multiplication by $|H|$ is an epimorphism and $\{x_1, \ldots, x_n\}$ is also a poor $M$-cosequence.

Proof. We first prove that the map is well defined by induction on $n$. Suppose that $n = 1$, and that $H = (h)$ with $y_1 = hx_1$. Then for all $m \in \text{Ann}_M(y_1)$, i.e., $my_1 = 0$, and $mhx_1 = 0$. Hence $|H|m \in \text{Ann}_M(x_1)$.

Assume that it is true when $n - 1$. Let $m \in \text{Ann}_M(y_1, \ldots, y_n)R$. Then we have $m \in \text{Ann}_M(y_1, \ldots, y_{n-1})R$ and $H'[x_1 \ldots x_{n-1}]^T = [y_1 \ldots y_{n-1}]^T$ where $H'$ is the top left $(n-1) \times (n-1)$ submatrix of $H$. Hence by inductive hypothesis we have

$$h_{11} \ldots h_{n-1,n-1}m \in \text{Ann}_M(x_1, \ldots, x_{n-1})R.$$

Since $m \in \text{Ann}_M(y_n)$, we get $my_n = m(\sum_{j=1}^{n} h_{nj}x_j) = 0$. Therefore we have

$$h_{11} \ldots h_{n-1,n-1}m(\sum_{j=1}^{n-1} h_{nj}x_j + h_{nn}x_n) = 0$$

or

$$h_{11} \ldots h_{nn}mx_n = 0.$$

Hence

$$|H|m \in \text{Ann}_M(x_1, \ldots, x_{n-1})R \cap \text{Ann}_M(x_n) = \text{Ann}_M(x_1, \ldots, x_n)R.$$

Now, we consider the following exact sequence;

$$\text{Ann}_M(y_1, \ldots, y_n)R \xrightarrow{|H|} \text{Ann}_M(x_1, \ldots, x_n)R \rightarrow C \rightarrow 0,$$

so that

$$\text{Hom}(R/(y_1, \ldots, y_n)R, M) \xrightarrow{|H|} \text{Hom}(R/(x_1, \ldots, x_n)R, M) \rightarrow C \rightarrow 0.$$
Hence we have the following exact sequence;

$$0 \rightarrow \text{Hom}(C, E) \rightarrow \text{Hom}(\text{Hom}(R/(x_1, \ldots, x_n)R, M), E)$$

$$\rightarrow \text{Hom}(\text{Hom}(R/(y_1, \ldots, y_n)R, M), E).$$

By Lemma 1.1(1), we obtain

$$0 \rightarrow \text{Hom}(C, E) \rightarrow R/(x_1, \ldots, x_n)R \otimes_R M^*$$

$$\rightarrow R/(y_1, \ldots, y_n)R \otimes_R M^*.$$ 

That is,

$$0 \rightarrow \text{Hom}(C, E) \rightarrow M^*/(x_1, \ldots, x_n)M^* \rightarrow M^*/(y_1, \ldots, y_n)M^*.$$ 

Since \{y_1, \ldots, y_n\} is a poor \(M^*\)-sequence by Lemma 1.2(2), we have \(\text{Hom}(C, E) = 0\) and \{x_1, \ldots, x_n\} is a poor \(M^*\)-sequence by Lemma 2.1. Hence we get \(C = 0\), since \(\text{Hom}(-, E)\) is faithfully exact.

**Corollary 2.3.** (cf. [M_t], 8) (1) If \(\{x_1, \ldots, x_n\}\) is a poor \(M\)-sequence, then

$$\alpha_t : M/(x_1^t, \ldots, x_n^t)M \xrightarrow{\varphi} M/(x_1^{t+1}, \ldots, x_n^{t+1})M$$

defined by \(\alpha_t(m + (x_1^t, \ldots, x_n^t)M) = \varphi(m + (x_1^{t+1}, \ldots, x_n^{t+1})M\) with \(\varphi = x_1 \cdots x_n\) is a monomorphism for all \(t > 0\).

(2) If \(\{x_1, \ldots, x_n\}\) is a poor \(M\)-cosequence, then

$$\beta_t : \text{Ann}_M(x_1^{t+1}, \ldots, x_n^{t+1})R \xrightarrow{\varphi} \text{Ann}_M(x_1^t, \ldots, x_n^t)R$$

induced by multiplication by \(\varphi = x_1 \cdots x_n\) is an epimorphism for all \(t > 0\).

**Proof.** From Lemma 1.2(3)(4), we have \(\{x_1^{\alpha_1}, \ldots, x_n^{\alpha_n}\}\) is a poor \(M\)-sequence (-cosequence) for any positive integers \(\alpha_1, \ldots, \alpha_n\).

Hence consider that \(H\) is a diagonal matrix \(\text{diag}(x_1, \ldots, x_n)\) such that

$$H[x_1^t \ldots x_n^t]^T = [x_1^{t+1} \ldots x_n^{t+1}]^T.$$ 

Then the corollary follows easily from Lemma 2.1 (Theorem 2.2).
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References


Department of Mathematics, Chungnam National University, Taejon 305-764, Korea