STABILITY OF ISOMETRIES BETWEEN
FINITE DIMENSIONAL HILBERT SPACES

KIL-WOUNG JUN AND DAL-WON PARK*

1. Introduction

It is a well-known classical result of Mazur and Ulam that an isometry \( T \) from a real Banach space \( X \) onto a real Banach space \( Y \) with \( T(0) = 0 \) is automatically linear[5]. A map \( T \) between Banach spaces \( X \) and \( Y \) is called an \( \epsilon \)-bi-Lipschitz map if

\[
(1 - \epsilon) \|x - y\| \leq \|Tx - Ty\| \leq (1 + \epsilon) \|x - y\| \text{ for } x, y \in X.
\]

Jarosz[3] conjectured that if \( X, Y \) are real Banach spaces such that there is a surjective \( \epsilon \)-bi-Lipschitz map between \( X \) and \( Y \), then \( X \) and \( Y \) are linearly isomorphic for sufficiently small \( \epsilon \).

The above statement is known to be true for certain special classes of Banach spaces like uniform algebras [2]. It is also known that this is false, even for \( C(K) \) spaces, if we do not assume that \( \epsilon \) is close to zero[1]. Mankiewicz [4] proved that if there is a surjective \( \epsilon \)-bi-Lipschitz map between a Banach space \( X \) and a Hilbert space \( Y \), then \( X \) and \( Y \) are linearly homeomorphic. In this note we show that if \( T \) is an \( \epsilon \)-bi-Lipschitz map from a Hilbert space \( X \) onto a Hilbert space \( Y \) with \( \dim X < \infty \), then there is an isometry from \( X \) onto \( Y \) which is near \( T \).

2. The result

**Theorem.** Let \( X \) and \( Y \) be real Hilbert spaces with \( \dim X < \infty \). If \( T \) is an \( \epsilon \)-bi-Lipschitz map from \( X \) onto \( Y \) with \( T(0) = 0 \) and with \( \epsilon \leq \epsilon_0 \), then there is an isometry \( I \) from \( X \) onto \( Y \) for which \( \|Tx - Ix\| \leq \epsilon_0 \).

Received November 1, 1993.
1991 Mathematics Subject Classification: Primary 46 E 20
Key words and Phrases: Hilbert space, \( \epsilon \)-bi-Lipschitz map
* This work was partially supported by KOSEF, Grant No 91-08-00-01.
\( C(\epsilon)(\|x\|^{\frac{1}{3}} + \|x\|^{\frac{3}{2}}) \) where \( \epsilon_0 \) is an absolute constant and \( C(\epsilon) \to 0 \) as \( \epsilon \to 0 \).

**Proof.** We divide the proof into a number of simple steps and at various points of the proof we use inequalities involving \( \epsilon \) which are valid only if \( \epsilon \) is sufficiently small; in these circumstances we will merely assume that \( \epsilon \) is near zero. Let \( \epsilon_1, \epsilon_2, \ldots, \epsilon_n \) be an orthonormal basis of \( X \). We denote the inner product in \( X \) and \( Y \) by \((,\)\).

**Step 1.**

\[
-6\sqrt{2}\epsilon - 18\epsilon^2 \leq \left( \frac{T\epsilon_i}{\|T\epsilon_i\|}, \frac{T\epsilon_j}{\|T\epsilon_j\|} \right) \leq 6\sqrt{2}\epsilon - 18\epsilon^2
\]

for \( i \neq j, i, j = 1, 2, \ldots, n \).

**Proof.** Let \( i \neq j \) and \( i, j = 1, 2, \ldots, n \). Since \( T \) is an \( \epsilon \)-bi-Lipschitz map, \( 1 - \epsilon \leq \|T\epsilon_i\| \leq 1 + \epsilon \) and

\[
\sqrt{2}(1 - \epsilon) \leq \|T\epsilon_i - T\epsilon_j\| \leq \sqrt{2}(1 + \epsilon).
\]

Thus we get

\[
\frac{\|T\epsilon_i\|}{\|T\epsilon_j\|} - \frac{T\epsilon_j}{\|T\epsilon_j\|} \leq \frac{1}{\|T\epsilon_j\|}(2\epsilon + (1 + \epsilon)\sqrt{2})
\]

\[
\leq \sqrt{2} + \frac{(2 + 2\sqrt{2})\epsilon}{1 - \epsilon}
\]

\[
\leq \sqrt{2} + 6\epsilon.
\]

Also, we have

\[
\frac{\|T\epsilon_i\|}{\|T\epsilon_j\|} - \frac{T\epsilon_j}{\|T\epsilon_j\|} \geq \frac{1}{\|T\epsilon_j\|} \left( \|T\epsilon_i - T\epsilon_j\| - \|T\epsilon_j\| - \|T\epsilon_i\| \right)
\]

\[
\geq \frac{1}{1 + \epsilon}((1 - \epsilon)\sqrt{2} - 2\epsilon)
\]

\[
\geq \sqrt{2} - 6\epsilon.
\]

By the above two inequalities, we have

\[
\sqrt{2} - 6\epsilon \leq \left( \frac{T\epsilon_i}{\|T\epsilon_i\|}, \frac{T\epsilon_j}{\|T\epsilon_j\|} \right) \leq \sqrt{2} + 6\epsilon.
\]
Hence
\[-6\sqrt{2}\epsilon - 18\epsilon^2 \leq \left( \frac{T_{e_i}}{\|Te_i\|}, \frac{T_{e_j}}{\|Te_j\|} \right) \leq 6\sqrt{2}\epsilon - 18\epsilon^2.\]

**STEP 2.** There is an orthonormal basis \( f_1, f_2, ..., f_n \) of \( Y \) for which
\[\| f_i - \frac{T_{e_i}}{\|Te_i\|} \| \leq C_i(\epsilon), i = 1, 2, ..., n\]
where \( C_i(\epsilon) \to 0 \) as \( \epsilon \to 0 \).

**Proof.** Put \( f_1 = \frac{T_{e_1}}{\|Te_1\|} \). Then \( C_1(\epsilon) = 0 \). Suppose \( f_1, ..., f_m(m < n) \) are linearly independent such that \( \|f_i\| = 1, (f_i, f_j) = 0 \) for \( i \neq j, i, j = 1, ..., m \) and
\[\| f_i - \frac{T_{e_i}}{\|Te_i\|} \| \leq C_i(\epsilon), i = 1, 2, ..., m.\]

Let \( S\epsilon_{m+1} = \{ f \in Y | \|f\| = 1, (f, f_1) = \cdots = (f, f_m) = 0, \frac{T_{\epsilon_{m+1}}}{\|Te_{m+1}\|} = \alpha_1 f_1 + \cdots + \alpha_m f_m + \beta f, \ \alpha_1, \alpha_2, ..., \alpha_m, \beta \text{ are real numbers} \} \). Suppose that \( S\epsilon_{m+1} \) is empty. Without loss of generality we can assume that \( \frac{T_{\epsilon_{m+1}}}{\|Te_{m+1}\|} = \alpha_1 f_1 + \cdots + \alpha_m f_m \). Then, by Schwarz inequality and Step 1,
\[\left| \left( \frac{T_{\epsilon_{m+1}}}{\|Te_{m+1}\|}, f_j \right) \right| \leq 6\sqrt{2}\epsilon + 18\epsilon^2 + C_j(\epsilon).\]
Thus we have
\[\left( \frac{T_{\epsilon_{m+1}}}{\|Te_{m+1}\|}, \frac{T_{\epsilon_{m+1}}}{\|Te_{m+1}\|} \right) \leq (|\alpha_1| + \cdots + |\alpha_m|)(6\sqrt{2}\epsilon + 18\epsilon^2 + C_1(\epsilon) + \cdots + C_m(\epsilon)) < m(6\sqrt{2}\epsilon + 18\epsilon^2 + C_1(\epsilon) + \cdots + C_m(\epsilon)) < 1.\]

This contradicts that \( \| \frac{T_{\epsilon_{m+1}}}{\|Te_{m+1}\|} \| = 1 \). Thus \( S\epsilon_{m+1} \neq \emptyset \). We choose a \( f \in S\epsilon_{m+1} \) and let \( f_{m+1} = f \). Thus \( \dim X \leq \dim Y \). Since \( T \)
is an $\epsilon$-bi-Lipschitz map, $\dim Y \leq \dim X$. Hence $f_1, f_2, \ldots, f_n$ is an orthonormal basis of $Y$. Since \( \frac{T_{n+1} e_{m+1}}{\|T_{n+1} e_{m+1}\|} = \alpha_1 f_1 + \cdots + \alpha_m f_m + \beta f_{m+1} \),

\[
\left\| f_{m+1} - \frac{T_{n+1} e_{m+1}}{\|T_{n+1} e_{m+1}\|} \right\|^2 = 2 - 2\beta.
\]

Since \( (\frac{T_{n+1} e_{m+1}}{\|T_{n+1} e_{m+1}\|}, f_i) = \alpha_i, i = 1, 2, \ldots, m \), we have \( |\alpha_i| \leq 6\sqrt{2}\epsilon + 18\epsilon^2 + C_1(\epsilon) \). Thus

\[
\beta^2 = 1 - \alpha_1^2 - \alpha_2^2 - \cdots - \alpha_m^2 \\
\geq 1 - m(6\sqrt{2}\epsilon + 18\epsilon^2) - (C_1(\epsilon) + C_2(\epsilon) + \cdots + C_m(\epsilon)).
\]

Hence

\[
2 - 2\beta \leq 2 - 2\beta^2 \\
\leq 2m(6\sqrt{2}\epsilon + 18\epsilon^2) + 2(C_1(\epsilon) + C_2(\epsilon) + \cdots + C_m(\epsilon)).
\]

Let \( C_{m+1}(\epsilon) = \sqrt{2m(6\sqrt{2}\epsilon + 18\epsilon^2) + 2(C_1(\epsilon) + C_2(\epsilon) + \cdots + C_m(\epsilon))} \). Then \( C_{m+1}(\epsilon) \rightarrow 0 \) as \( \epsilon \rightarrow 0 \).

**Step 3.** \( \|\lambda Tx - T\lambda x\| \leq 4\sqrt{\epsilon}(|\lambda|^\frac{1}{2} + |\lambda|^\frac{3}{2})\|x\| \) for \( \lambda \in \mathbb{R} \) and \( \|Tx + Ty - T(x + y)\| \leq 4\sqrt{\epsilon} (\|x\| + \|y\|) \).

**Proof.** Since \( T \) is an \( \epsilon \)-bi-Lipschitz map,

\[
(1 - \epsilon)|1 - \lambda|\|x\| \leq \|Tx - T\lambda x\| \leq (1 + \epsilon)|1 - \lambda|\|x\|.
\]

A routine calculation shows that

\[
(1 - \epsilon)^2(1 - \lambda)^2\|x\|^2 - \|Tx\|^2 - \|T\lambda x\|^2 \\
\leq -2\langle Tx, T\lambda x \rangle \\
\leq (1 + \epsilon)^2(1 - \lambda)^2\|x\|^2 - \|Tx\|^2 - \|T\lambda x\|^2.
\]

So we have

\[
\|\lambda Tx - T\lambda x\| \leq 4|\lambda|^\frac{3}{2}\sqrt{\epsilon}\|x\| \text{ for } \lambda \geq 1, \lambda \leq -1.
\]
Hence for $-1 < \lambda < 1$, we get

$$||\lambda Tx - T\lambda x|| \leq 4|\lambda|^\frac{1}{2} \sqrt{\epsilon} ||x||.$$ 

Thus for any real number $\lambda$, we get

$$||\lambda Tx - T\lambda x|| \leq 4\sqrt{\epsilon}(|\lambda|^\frac{1}{2} + |\lambda|^\frac{3}{2}) ||x||.$$ 

It is easy to see that

$$ (1 - \epsilon)^2 ||y||^2 - ||Tx||^2 - ||T(x + y)||^2 $$

$$ \leq -2(Tx, T(x + y)) $$

$$ \leq (1 + \epsilon)^2 ||y||^2 - ||Tx||^2 - ||T(x + y)||^2, $$

$$ (1 - \epsilon)^2 ||x||^2 - ||Ty||^2 - ||T(x + y)||^2 $$

$$ \leq -2(Ty, T(x + y)) $$

$$ \leq (1 + \epsilon)^2 ||x||^2 - ||Ty||^2 - ||T(x + y)||^2 $$

and

$$ (1 - \epsilon)^2 ||x - y||^2 - ||Tx||^2 - ||Ty||^2 $$

$$ \leq -2(Tx, Ty) $$

$$ \leq (1 + \epsilon)^2 ||x - y||^2 - ||Tx||^2 - ||Ty||^2. $$

So we have

$$ ||Tx + Ty - T(x + y)|| \leq 4\sqrt{\epsilon}(||x|| + ||y||). $$

**Step 4.** There is an isometry $I$ from $X$ onto $Y$ for which $||Ix - Tx|| \leq C(\epsilon)(||x||^\frac{1}{2} + ||x||^\frac{3}{2})$ where $C(\epsilon) \to 0$ as $\epsilon \to 0$.

**Proof.** For $x \in X$, there are $\alpha_1, ..., \alpha_n$ such that $x = \alpha_1 e_1 + \cdots + \alpha_n e_n$. We define $I : X \longrightarrow Y$ by $Ix = \alpha_1 f_1 + \alpha_2 f_2 + \cdots + \alpha_n f_n$. Then $I$ is an isometry. By Step 3, we have

$$ (1) \ |T(\alpha_1 e_1 + \cdots + \alpha_n e_n) - T(\alpha_1 e_1) - \cdots - T(\alpha_n e_n)| \leq 8n\sqrt{\epsilon} ||x||. $$
By Step 2, we get

$$
\|Te_i - f_i\| \leq \left\| Te_i - \frac{Te_i}{\|Te_i\|} \right\| + \left\| \frac{Te_i}{\|Te_i\|} - f_i \right\| \leq \epsilon + C_i(\epsilon) \text{ for } i = 1, 2, \ldots, n.
$$

By (1), (2) and Step 3, we obtain

$$
\|Tx - Ix\| \leq 8n \sqrt{\epsilon} \|x\| + 4n \sqrt{\epsilon} \left( \|x\|^{\frac{1}{2}} + \|x\|^{\frac{3}{2}} \right) + (n \epsilon + C_1(\epsilon) + \cdots + C_n(\epsilon))n\|x\|.
$$

Thus we have

$$
\|Tx - Ix\| \leq C(\epsilon)(\|x\|^{\frac{1}{2}} + \|x\|^{\frac{3}{2}})
$$

where $C(\epsilon) = 12n \sqrt{\epsilon} + n^2 \epsilon + n(C_1(\epsilon) + \cdots + C_n(\epsilon))$. This completes the proof of Step 4.

References


Kil-Woung Jun
Department of Mathematics
Chungnam National University
Taejon 305-764, Korea

Dal-Won Park
Department of Mathematics Education
Kongju National University
Kongju 314-701, Korea