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A RELATIVE NIELSEN NUMBER IN COINCIDENCE THEORY

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1. Introcuction.

Nielsen coincidence theory is concerned with the estimation of a lower bound for the number of coincidences of two maps $f, g: X \longrightarrow Y$. For this purpose the so-called Nielsen number $N(f,g)$ is introduced, which is a lower bound for the number of coincidences ([1]). The relative Nielsen number $N(f : X, A)$ in the fixed point theory is introduced in [3], which is a lower bound for the number of fixed points for all maps in the relative homotopy class of $f:(X,A) \longrightarrow (X,A)$, and its estimation is given in [5].

It is the purpose of this paper to define relative Nielsen number $N_R(f,g)$ for maps $f,g:(X,A) \longrightarrow (Y,B)$ of pair of connected compact polyhedra and give its estimation in the same way as in [3] and [5].

2. Some properties about liftings.

In this section we will give a few facts about liftings in coincidence theory by generalizing those in fixed point theory ([2;Cahpter III]). The spaces considered are assumed to have universal covering spaces. The universal covering will be denoted by $p_X : \tilde{X} \longrightarrow X$.

PROPOSITION 2.1. *Let f, f', h, h' be maps such that*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h'} \\ X' & \xrightarrow[f']{} & Y' \end{array}$$

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commutes. Given a lifting $\tilde{f}$ of f, a lifting $\tilde{h}$ of h, and a lifting $\tilde{h}'$ of h', there is a unique lifting $\tilde{f}'$ of f' such that

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{f}} & \tilde{Y} \\ {\scriptstyle \tilde{h}}\downarrow & & \downarrow{\scriptstyle \tilde{h}'} \\ \tilde{X}' & \xrightarrow[\tilde{f}']{} & \tilde{Y}' \end{array}$$

commutes.

Proof. See [2; Ch. 3, proposition 1.5] □

A lifting pair $(\tilde{h}, \tilde{h}')$ of h, h' determines a correspondence $(\tilde{h}, \tilde{h}')_{\text{lift}}$ from liftings of f to liftings of f' by the above proposition. Given the following diagrams,

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h'} \\ X' & \xrightarrow[f']{} & Y' \end{array} \qquad\qquad \begin{array}{ccc} X & \xrightarrow{g} & Y \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h'} \\ X' & \xrightarrow[g']{} & Y' \end{array}$$

the correspondence $(\tilde{h}, \tilde{h}')_{\text{lift}}$ gives rise to a correspondence $\overline{(\tilde{h}, \tilde{h}')_{\text{lift}}}$ from lifting pairs of f, g to lifting pairs of f', g' ;

$$\overline{(\tilde{h}, \tilde{h}')_{\text{lift}}} : (\tilde{f}, \tilde{g}) \longmapsto \Big((\tilde{h}, \tilde{h}')_{\text{lift}}(\tilde{f}), (\tilde{h}, \tilde{h}')_{\text{lift}}(\tilde{g}) \Big),$$

where $(\tilde{f}, \tilde{g})$ is a lifting pair of f, g. In particular, the correspondence $(\tilde{h}, \tilde{h})_{\text{lift}}$ from liftings of the identity map of X to liftings of the the identity map of X' is the homomorphism $\tilde{h}_\pi$ from $\pi_1(X)$ to $\pi_1(X')$ determined the lifting $\tilde{h}$ of h, both identified with the groups of covering translations. (See [4, p. 16]).

THEOREM 2.2. (i) *The correspondence $\overline{(\tilde{h}, \tilde{h}')_{\text{lift}}}$ sends a lifting class of f and g into a lifting class of f' and g', that is,*

$$\overline{(\tilde{h}, \tilde{h}')_{\text{lift}}}[\tilde{f}, \tilde{g}] \subset [(\tilde{h}, \tilde{h}')_{\text{lift}}(\tilde{f}), (\tilde{h}, \tilde{h}')_{\text{lift}}(\tilde{g})].$$

(ii) *On the lifting class level, this correspondence does not depend on the lifting pair* $(\tilde{h}, \tilde{h}')$ *of* h *and* h' *but is determined by* h *and* h' *themselves.*

Proof. (i) $(\tilde{h}, \tilde{h}')_{\text{lift}}(\alpha \circ \tilde{f} \circ \beta) = \tilde{h}'_\pi(\alpha) \circ (\tilde{h}, \tilde{h}')_{\text{lift}}(\tilde{f}) \circ \tilde{h}_\pi(\beta)$, where $\alpha \in \pi_1(Y)$ and $\beta \in \pi_1(X)$.

(ii) If $\tilde{h}_1 = \alpha \circ \tilde{h}$ and $\tilde{h}'_1 = \beta \circ \tilde{h}'$ are another liftings of h and h', respectively, where $\alpha \in \pi_1(X')$ and $\beta \in \pi_1(Y')$ and $(\tilde{h}, \tilde{h}')_{\text{lift}}: \tilde{f} \longmapsto \tilde{f}'$, then

$$(\tilde{h}_1, \tilde{h}'_1)_{\text{lift}}: \tilde{f} \longmapsto \beta \circ \tilde{f}' \circ \alpha^{-1}. \qquad \square$$

By the above theorem, $(\tilde{h}, \tilde{h}')_{\text{lift}}$ induces a correspondence from lifting classes of f, g to lifting classes of f', g', which is independent of the choice of the lifting pair of h, h' and is determined by h and h' themselves. It is denoted

$$(h, h')_C: C(f, g) \longrightarrow C(f', g')$$

where $C(f, g)$ is the set of all lifting classes of f and g.

Recall that a coincidence class is always lebelled by a lifting class, and we have

PROPOSITION 2.3. *Every coincidence class of* $f, g: X \longrightarrow Y$ *is mapp ed by* h *into some coincidence class of* $f', g': X' \longrightarrow Y'$. *Namely, if*

$$(h, h')_C: [\tilde{f}, \tilde{g}] \longmapsto [\tilde{f}', \tilde{g}'],$$

then $hp_X\Gamma(\tilde{f}, \tilde{g}) \subset p_{X'}\Gamma(\tilde{f}', \tilde{g}')$.

Proof. If $\overline{(\tilde{h}, \tilde{h}')}_{\text{lift}}: (\tilde{f}, \tilde{g}) \longmapsto (\tilde{f}', \tilde{g}')$, then $\tilde{h}\Gamma(\tilde{f}, \tilde{g}) \subset \Gamma(\tilde{f}', \tilde{g}')$. $\square$

PROPOSITION 2.4. *Let* $f, g: X \longrightarrow Y$ *and* $f', g': X' \longrightarrow Y'$ *be maps such that* $h' \circ f = f' \circ h$ *and* $h' \circ g = g' \circ h$. *If* $h'_\pi: \pi_1(Y) \longrightarrow \pi_1(Y')$ *is surjective, then* $(\tilde{h}, \tilde{h}')_{\text{lift}}$ *is surjective, hence* $\overline{(\tilde{h}, \tilde{h}')}_{\text{lift}}$ *is also surjective and so is* $(h, h')_C$.

Proof. Suppose $(\tilde{h}, \tilde{h}')_{\text{lift}}: \tilde{f}_0 \longmapsto \tilde{f}'_0$, where $\tilde{f}_0, \tilde{f}'_0$ are liftings of f, f', respectively. Then for any liftings $\tilde{f}'$ of f', we have $\tilde{f}' = \beta \circ \tilde{f}'_0$ for some $\beta \in \pi_1(Y')$. Since h'_π is surjective, $\tilde{h}'_\pi$ is surjective (See p. 45, Lemma 1.13 [2]). Thus there is $\alpha \in \pi_1(Y)$ such that $\tilde{h}'_\pi(\alpha) = \beta$ and $(\tilde{h}, \tilde{h}')_{\text{lift}}: \alpha \circ \tilde{f}_0 \longmapsto \beta \circ \tilde{f}'_0$. $\square$

PROPOSITION 2.5. *Let* $\{f_t\}_{t\in I}: X \longrightarrow Y$, $\{f'_t\}_{t\in I}: X' \longrightarrow Y'$, $\{h_t\}_{t\in I}: X \longrightarrow X'$, $\{h'_t\}_{t\in I}: Y \longrightarrow Y'$ *be homotopies such that* $h'_t \circ f_t = f'_t \circ h_t$ *for all* $t \in I$. *Let* $\{\tilde{f}_t\}_{t\in I}$ *be a lifting of* $\{f_t\}_{t\in I}$, $\{\tilde{h}_t\}_{t\in I}$ *a lifting of* $\{h_t\}_{t\in I}$, *and* $\{\tilde{h}'_t\}_{t\in I}$ *a lifting of* $\{h'_t\}_{t\in I}$. *Then there is a unique lifting* $\{\tilde{f}'_t\}_{t\in I}$ *of* $\{f'_t\}_{t\in I}$ *such that* $\tilde{h}'_t \circ \tilde{f}_t = \tilde{f}'_t \circ \tilde{h}_t$ *for all* $t \in I$. *Hence if* $\{g_t\}_{t\in I}: X \longrightarrow Y$ *and* $\{g'_t\}_{t\in I}: X' \longrightarrow Y'$ *are homotopies such that* $h'_t \circ g_t = g'_t \circ h_t$ *for all* $t \in I$ *and* $\{\tilde{g}_t\}_{t\in I}$ *is a lifting of* $\{g_t\}_{t\in I}$, *then the diagram*

$$\begin{array}{ccc} [\tilde{f}_0, \tilde{g}_0] & \xrightarrow{(\{f_t\},\{g_t\})} & [\tilde{f}_1, \tilde{g}_1] \\ {\scriptstyle (h_0,h'_0)_C}\downarrow & & \downarrow{\scriptstyle (h_1,h'_1)_C} \\ [\tilde{f}'_0, \tilde{g}'_0] & \xrightarrow{(\{f'_t\},\{g'_t\})} & [\tilde{f}'_1, \tilde{g}'_1] \end{array}$$

commutes, where $\{\tilde{g}'_t\}$ *is the unique lifting of* $\{g'_t\}$ *such that* $\tilde{g}'_t \circ \tilde{h}_t = \tilde{h}'_t \circ \tilde{g}_t$ *for all* $t \in I$.

3. The Nielsen number.

DEFINITION 3.1. A triple (f, g, A) is *admissible* if $f, g: X \longrightarrow Y$ are maps, $A \subset X$, and there is a closed set $N \subset X$ with $\mathrm{Cl}\, A \subset \mathrm{Int}\, N$ and $\Gamma(f,g) \cap (N - A)$ empty. A function w from the admissible triples into an abelian group is *a coincidence index* if it satisfies the following two conditions :

1. (Additivity) If $A \subset X$ and $\{A_i\}$ is a finite indexed collection of subsets of A such that
   (i) (f, g, A) is admissible, and (f, g, A_i) is admissible for each i, and
   (ii) $(A - \bigcup_i A_i) \cap \Gamma(f,g)$ is empty,

   then

$$w(f,g,A) = \sum_i w(f,g,A_i).$$

2. (Homotopy) If

$$\{f_t\}_{t\in I}, \ \{g_t\}_{t\in I}: X \longrightarrow Y$$

are homotopies, $A \subset X$ is open, and (f_t, g_t, A) is adimissible for each $t \in I$, then

$$w(f_0, g_0, A) = w(f_1, g_1, A).$$

PROPOSITION 3.2. *Suppose that $f, g: X \longrightarrow Y$ are maps. Then $(f, g, p_X\Gamma(\tilde{f}, \tilde{g}))$ is admissible for each coincidence class $p_X\Gamma(\tilde{f}, \tilde{g})$ of f and g, and*

$$w(f, g, X) = \sum_{[\tilde{f}, \tilde{g}]} w(f, g, p_X\Gamma(\tilde{f}, \tilde{g})).$$

Proof. Since the index of empty set is zero, it follows from [1, IV, proposition 21]. □

DEFINITION 3.3. A coincidence class $p_X\Gamma(\tilde{f}, \tilde{g})$ of f and g is said to be *essential* if $w(f, g, p_X\Gamma(\tilde{f}, \tilde{g})) \neq 0$. *The Nielsen number* $N(f, g)$ of the maps $f, g: X \longrightarrow Y$ is defined to be the number of essential coincidence classes of f and g.

PROPOSITION 3.4. *Let $F = \{ f_t \}_{t \in I}: X \longrightarrow Y$, $G = \{ g_t \}_{t \in I}: X \longrightarrow Y$ be homotopies, and let $(\tilde{f}_i, \tilde{g}_i)$ be a lifting pair of $f_i, g_i, i = 0, 1$. If a class $p_X\Gamma(\tilde{f}_0, \tilde{g}_0)$ corresponds to a class $p_X\Gamma(\tilde{f}_1, \tilde{g}_1)$ via F, G, then*

$$w(f_0, g_0, p_X\Gamma(\tilde{f}_0, \tilde{g}_0)) = w(f_1, g_1, p_X\Gamma(\tilde{f}_1, \tilde{g}_1)).$$

Proof. It follows from [4, Theorem 2.5], [1, IV, proposition 21] and the index of empty set is zero. □

The number of coincidence classes is a homotopy invariant ([4, Theorem 2.5] and by proposition 3.4, we have

COROLLARY 3.5. *Let $f_0, f_1: X \longrightarrow Y$ are homotopic and let $g_0, g_1: X \longrightarrow Y$ are homotopic. Then*

$$N(f_0, g_0) = N(f_1, g_1).$$

4. The relative Nielsen number

We always assume $f, g\colon (X, A) \longrightarrow (Y, B)$ be two maps of pairs of connected compact polyhedras. We shall write $\bar{f}\colon A \longrightarrow B$ for the restriction of f to A and write $f\colon X \longrightarrow Y$ if the condition that $f(A) \subset B$ is immaterial, and homotopies of $f\colon (X, A) \longrightarrow (Y, B)$ are maps of the form $\{ f_t, \bar{f}_t \}_{t \in I}\colon (X, A) \longrightarrow (Y, B)$ where $\bar{f}_t = f_t|_A$ for all $t \in I$. Now the diagrams

$$\begin{array}{ccc} A & \xrightarrow{\bar{f}} & B \\ {\scriptstyle i'}\downarrow & & \downarrow{\scriptstyle i} \\ X & \xrightarrow[f]{} & Y \end{array} \qquad\qquad \begin{array}{ccc} A & \xrightarrow{\bar{g}} & B \\ {\scriptstyle i'}\downarrow & & \downarrow{\scriptstyle i} \\ X & \xrightarrow[g]{} & Y \end{array}$$

commute, where i' and i are inclusions. The spaces X, Y, A, B have universal coverings

$$p\colon \tilde{X} \longrightarrow X \qquad \bar{p}\colon \tilde{A} \longrightarrow A$$
$$q\colon \tilde{Y} \longrightarrow Y \qquad \bar{q}\colon \tilde{B} \longrightarrow B$$

DEFINITION 4.1. A coincidence class $p\Gamma(\tilde{f}, \tilde{g})$ of f and g is *a common coincidence class of f and g* if $(i, i')_C[\tilde{\bar{f}}, \tilde{\bar{g}}] = [\tilde{f}, \tilde{g}]$ for some lifting pair $(\tilde{\bar{f}}, \tilde{\bar{g}})$ of $\bar{f}, \bar{g}$. In this case, if $p\Gamma(\tilde{f}, \tilde{g})$ and $p\Gamma(\tilde{\bar{f}}, \tilde{\bar{g}})$ are essential, then the class $p\Gamma(\tilde{f}, \tilde{g})$ is called *an essential common coincidence class of f and g. The relative Nielsen number $N_R(f, g)$ of* $f, g\colon (X, A) \longrightarrow (Y, B)$ is defined as

$$N_R(f, g) = N(f, g) + N(\bar{f}, \bar{g}) - N(f, g; \bar{f}, \bar{g}),$$

where $N(f, g; \bar{f}, \bar{g})$ is the number of essential common coincidence classes of f and g. It is obvious from the definition that $N_R(f, g)$ is a lower bound for the number of coincidences of f and g satisfying $N_R(f, g) \geq N(f, g)$ and $N_R(f, g) \geq N(\bar{f}, \bar{g})$.

THEOREM 4.2. *Let $f, g\colon (X, A) \longrightarrow (Y, B)$ be two maps of pairs of connected compact polyhedras. Then*

(i) *If $N(f, g) = 0$, then $N_R(f, g) = N(\bar{f}, \bar{g})$.*
(ii) *If $N(\bar{f}, \bar{g}) = 0$, then $N_R(f, g) = N(f, g)$.*
(iii) *If Y is simply connected or $f, g\colon (X, A) \longrightarrow (Y, B)$ are homotopic, and $N(\bar{f}, \bar{g}) \neq 0$, then $N_R(f, g) = N(\bar{f}, \bar{g})$.*

Proof. (i) and (ii) are obvious from the definition, ae in both cases $N(f, g; \bar{f}, \bar{g}) = 0$. We only have to consider the case $N(f, g) \neq 0$ in (iii). If Y is simply connected, then $f, g\colon X \longrightarrow Y$ has only one essential coincidence class $p\Gamma(\tilde{f}, \tilde{g})$, and each essential coincidence class of $\bar{f}$ and $\bar{g}$ belongs to $p\Gamma(\tilde{f}, \tilde{g})$ by proposition 2.3, so $N(f, g; \bar{f}, \bar{g}) = 1$. If $f, g\colon (X, A) \longrightarrow (Y, B)$ are homotopic, then the same argument applies. □

THEOREM 4.3. *If $f_0, f_1\colon (X, A) \longrightarrow (Y, B)$ are homotopic and $g_0, g_1\colon (X, A) \longrightarrow (Y, B)$ are homotopic, then $N_R(f_0, g_0) = N_R(f_1, g_1)$.*

Proof. Let $\{f_t, \bar{f}_t\}, \{g_t, \bar{g}_t\}\colon (X, A) \longrightarrow (Y, B)$ are homotopies from f_0 to f_1 and from g_0 to g_1, respectively. There exists an index-preserving bijection $(\{f_t\}, \{g_t\})\colon C(f_0, g_0) \longrightarrow C(f_1, g_1)$. As it follows from corollary 3.5 that $N(f_0, g_0) = N(f_1, g_1)$ and $N(\bar{f}_0, \bar{g}_0) = N(\bar{f}_1, \bar{g}_1)$, it suffices to show that $(\{f_t\}, \{g_t\})$sends an essential common coincidence class of f_0 and g_0 to an essential common coincidence class of f_1 and g_1. Let $p\Gamma(\tilde{f}_0, \tilde{g}_0)$ be an essential common coincidence class of f_0 and g_0, then there exists a lifting pair $(\tilde{\tilde{f}}_0, \tilde{\tilde{g}}_0)$ of $\bar{f}_0, \bar{g}_0$ such that $(i', i)_C[\tilde{\tilde{f}}_0, \tilde{\tilde{g}}_0] = [\tilde{f}_0, \tilde{g}_0]$ and $\bar{p}\Gamma(\tilde{\tilde{f}}_0, \tilde{\tilde{g}}_0)$ is essential. By proposition 2.5, if $(\{\bar{f}_t\}, \{\bar{g}_t\})$ sends $[\tilde{\tilde{f}}_0, \tilde{\tilde{g}}_0]$ to $[\tilde{\tilde{f}}_1, \tilde{\tilde{g}}_1]$ then we have a commutative diagram

$$\begin{array}{ccc}
[\tilde{\tilde{f}}_0, \tilde{\tilde{g}}_0] & \xrightarrow{(\{\bar{f}_t\}, \{\bar{g}_t\})} & [\tilde{\tilde{f}}_1, \tilde{\tilde{g}}_1] \\
{\scriptstyle (i', i)_C}\downarrow & & \downarrow{\scriptstyle (i', i)_C} \\
[\tilde{f}_0, \tilde{g}_0] & \xrightarrow{(\{f_t\}, \{g_t\})} & [\tilde{f}_1, \tilde{g}_1]
\end{array}$$

Therefore $p\Gamma(\tilde{f}_1, \tilde{g}_1)$ is an essential common coincidence class of f_1 and g_1. □

5. Computation of $N_R(f,g)$

Suppose $x_0 \in A \subset X$ and $y_0 \in B \subset Y$ are base points such that $f(x_0) = g(x_0) = y_0$. Then, recall that points of universal covering spaces are identified with path classes in base spaces starting from base points. Under this identification, let $\tilde{a}_0 \in \bar{p}^{-1}(x_0) \subset \tilde{A}$, $\tilde{x}_0 \in p^{-1}(x_0) \subset \tilde{X}$, $\tilde{b}_0 \in \bar{q}^{-1}(y_0) \subset \tilde{B}$, $\tilde{y}_0 \in q^{-1}(y_0) \subset \tilde{Y}$ be the constant paths. Then there are unique lifting pairs $(\tilde{\bar{f}}, \tilde{\bar{g}})$ of $\bar{f}, \bar{g}$ and $(\tilde{f}, \tilde{g})$ of f, g such that $\tilde{\bar{f}}(\tilde{a}_0) = \tilde{\bar{g}}(\tilde{a}_0) = \tilde{b}_0$ and $\tilde{f}(\tilde{x}_0) = \tilde{g}(\tilde{x}_0) = \tilde{y}_0$. By [2, III, 1.13 Lemma], $\tilde{\bar{f}}_\pi = \bar{f}_\pi$, $\tilde{\bar{g}}_\pi = \bar{g}_\pi$, $\tilde{f}_\pi = f_\pi$, $\tilde{g}_\pi = g_\pi$.

Throughtout this section, the lifting pair $(\tilde{\bar{f}}, \tilde{\bar{g}})$ of $\bar{f}, \bar{g}$ and the lifting pair $(\tilde{f}, \tilde{g})$ of f, g are chosen as references.

Let $F, G: X \times I \longrightarrow Y$ be cyclic homotopies at f, g, respectively. Then the path $\langle F, x_0\rangle^{-1}\langle G, x_0\rangle$ is a loop in Y at y_0, where $\langle F, x_0\rangle$ is a path in Y defined by

$$\langle F, x_0\rangle(t) = F(x_0, t).$$

Thus $\left[\langle F, x_0\rangle^{-1}\langle G, x_0\rangle\right] \in \pi_1(Y, y_0)$. The set of all such elements of $\pi_1(Y, y_0)$ is denoted by

$$J(f, g, x_0).$$

PROPOSITION 5.1. *Suppose $J(f,g,x_0) = \pi_1(Y, y_0)$. Then any two coincidence classes of f and g have the same index. Thus if $w(f,g,X) \neq 0$, then $N(f,g) = R(f,g)$.*

Proof. For any $\alpha, \beta \in \pi_1(Y, y_0)$, we have $\alpha^{-1}\beta = \left[\langle F, x_0\rangle^{-1}\langle G, x_0\rangle\right]$. Then there exist liftings $\tilde{F}: \tilde{f} \simeq \alpha' \circ \tilde{f}$ and $\tilde{G}: \tilde{g} \simeq \beta' \circ \tilde{g}$ of F and G, respectively, and $\alpha' = [\langle F, x_0\rangle]$, $\beta' = [\langle G, x_0\rangle]$ so that $\alpha^{-1}\beta = \alpha'^{-1}\beta'$ (See the proof of Lemma 4.11 [4]). Thus by proposition 3.4, [4,Lemma 4.5] and [4,Theorem 1.5], we have $w(f, g, p_X\Gamma(\tilde{f}, \tilde{g})) = w(f, g, p_X\Gamma(\alpha \circ \tilde{f}, \beta \circ \tilde{g}))$. Since α and β are arbitrary, so any two coincidence classes have the same index. It follows from [4,Theorem 4.6] and proposition 3.2 that $w(f,g,X) \neq 0$ implies $N(f,g) = R(f,g)$. □

THEOREM 5.2. *Let $f, g: (X, A) \longrightarrow (Y, B)$ be maps of pairs of connected compact polyhedras. Suppose that B is a Jiang space and $i_\pi: \pi_1(B) \longrightarrow \pi_1(Y)$ is surjective. If $w(\bar{f}, \bar{g}, A) \neq 0$, then $N_R(f,g) =$*

$\#\mathrm{coker}(\bar g_* - \bar f_*)$, where $\bar f_*, \bar g_*: H_1(A) \longrightarrow H_1(B)$ are one dimensional homology homomorphisms induced by f and g with integer coefficients.

Proof. If B is a Jiang space, then $\pi_1(B)$ is abelian. Thus it follows from proposition 5.1 and [4, Corollary 4.16] that $w(\bar f, \bar g, A) \neq 0$ implies $N(\bar f, \bar g) = \#\mathrm{coker}(\bar g_* - \bar f_*)$. By proposition 2.4, $(i', i)_C$ is surjective; thus we have $N(f,g) = N(f,g;\bar f,\bar g)$ so that $N_R(f,g) = N(\bar f, \bar g)$ □

LEMMA 5.3. *There exist one-to-one correspondences*

$$\bar\phi: C(\bar f, \bar g) \longrightarrow \pi_1'(B, y_0)$$
$$\phi: C(f, g) \longrightarrow \pi_1'(Y, y_0)$$

defined by

$$\bar\phi[\bar\alpha \circ \tilde{\bar f}, \bar\beta \circ \tilde{\bar g}] = [\bar\alpha^{-1}\bar\beta]$$
$$\phi[\alpha \circ \tilde f, \beta \circ \tilde g] = [\alpha^{-1}\beta]$$

where $\bar\alpha, \bar\beta \in \pi_1(B, y_0)$, $\alpha, \beta \in \pi_1(Y, y_0)$, and $\pi_1'(Y, y_0)$ is the set of f_π, g_π-conjugate classes in $\pi_1(Y, y_0)$.

Proof. See [4, p. 19, Theorem 4.6]. □

If $\bar\alpha$ and $\bar\beta$ are $\bar f_\pi, \bar g_\pi$-conjugate classes in $\pi_1(B, y_0)$, then $i_\pi(\bar\alpha)$ and $i_\pi(\bar\beta)$ are f_π, g_π-conjugate classes in $\pi_1(Y, y_0)$. Thus the homomorphism $i_\pi: \pi_1(B, y_0) \longrightarrow \pi_1(Y, y_0)$ induces a transformation $\nu: \pi_1'(B, y_0) \longrightarrow \pi_1'(Y, y_0)$.

PROPOSITION 5.4. *A coincidence class of f and g is a common coincidence class if and only if it corresponds to an element in the image of ν.*

Proof. If $(\tilde i', \tilde i)$ is the lifting pair of i', i such that $\tilde i'(\tilde a_0) = \tilde x_0$ and $\tilde i(\tilde b_0) = \tilde y_0$, then $\tilde f \circ \tilde i'(\tilde a_0) = \tilde i \circ \tilde{\bar f}(\tilde a_0)$, and $\tilde f \circ \tilde i'$ and $\tilde i \circ \tilde{\bar f}$ are both liftings of the same map $i \circ \bar f = f \circ i'$. By the unique lifting property of covering spaces, we have $\tilde f \circ \tilde i' = \tilde i \circ \tilde{\bar f}$. Similarly, $\tilde g \circ \tilde i' = \tilde i \circ \tilde{\bar g}$. Thus $(i', i)_C[\tilde{\bar f}, \tilde{\bar g}] = [\tilde f, \tilde g]$. Now it suffices to show that the diagram

$$\begin{array}{ccc}
C(\bar f, \bar g) & \xrightarrow{(i',i)_C} & C(f, g) \\
\bar\phi \big\downarrow & & \big\downarrow \phi \\
\pi_1'(B, y_0) & \xrightarrow[\nu]{} & \pi_1'(Y, y_0)
\end{array}$$

commutes. Let $[\bar\alpha \circ \tilde{\bar f}, \bar\beta \circ \tilde{\bar g}] \in C(\bar f, \bar g)$, then

$$\begin{aligned} \phi \circ (i', i)_C[\bar\alpha \circ \tilde{\bar f}, \bar\beta \circ \tilde{\bar g}] &= \phi[\tilde{i}_\pi(\bar\alpha) \circ \tilde f, \tilde{i}_\pi(\bar\beta) \circ \tilde g] \\ &= [\tilde{i}_\pi(\bar\alpha^{-1}) \tilde{i}_\pi(\bar\beta)] \\ &= [i_\pi(\bar\alpha^{-1}\bar\beta)] \end{aligned}$$

and

$$\nu \circ \bar\phi[\bar\alpha \circ \tilde{\bar f}, \bar\beta \circ \tilde{\bar g}] = \nu[\bar\alpha^{-1}\bar\beta] = [i_\pi(\bar\alpha^{-1}\bar\beta)]. \qquad \square$$

Consider the commutative diagram

$$\begin{array}{ccccc} \pi_1(B, y_0) & \longrightarrow & H_1(B) & \longrightarrow & \operatorname{coker}(\bar g_* - \bar f_*: H_1(A) \to H_1(B)) \\ {\scriptstyle i_\pi}\downarrow & & \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle i_*} \\ \pi_1(Y, y_0) & \longrightarrow & H_1(Y) & \longrightarrow & \operatorname{coker}(g_* - f_*: H_1(X) \to H_1(Y)) \end{array}$$

where $\bar\theta$, θ are abelianization and $\bar\eta$, η are the natural projection. Then, by [4,Theorem 4.15], we have

LEMMA 5.5. *The composition $\bar\eta \circ \bar\theta$ and $\eta \circ \theta$ induce correspondences*

$$\bar\tau: \pi_1'(B, y_0) \longrightarrow \operatorname{coker}(\bar g_* - \bar f_*)$$
$$\tau: \pi_1'(Y, y_0) \longrightarrow \operatorname{coker}(g_* - f_*)$$

and the diagram

$$\begin{array}{ccc} \pi_1'(B, y_0) & \xrightarrow{\ \bar\tau\ } & \operatorname{coker}(\bar g_* - \bar f_*) \\ {\scriptstyle \nu}\downarrow & & \downarrow{\scriptstyle i_*} \\ \pi_1'(Y, y_0) & \xrightarrow[\ \tau\ ]{} & \operatorname{coker}(g_* - f_*) \end{array}$$

commutes.

THEOREM 5.6. *Let $f, g: (X, A) \longrightarrow (Y, B)$ be maps of pairs of connected compact polyhedras, and let Y and B be Jiang spaces. Then, if $w(\bar f, \bar g, A) \cdot w(f, g, X) \neq 0$, then*

$$N_R(f, g) = \#\operatorname{coker}(g_* - f_*) + \#\operatorname{coker}(\bar g_* - \bar f_*) - \#i_*\operatorname{coker}(\bar g_* - \bar f_*).$$

Proof. Since $\pi_1(B, y_0)$ and $\pi_1(Y, y_0)$ are abelian groups, by [4, Corollary 4.16], τ, $\bar\tau$ are bijective. Apply Proposition 5.4 and Lemma 5.5 to get the conclusion. $\square$

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