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A CHARACTERIZATION OF CROSSED PRODUCTS WITHOUT COHOMOLOGY

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1. Introduction

Let $\mathbf{N}$ be a II_1 factor and G be a finite group acting outerly on $\mathbf{N}$. Then the crossed product algebra $\mathbf{M} = \mathbf{N} \rtimes G$ is also a II_1 factor and $\mathbf{N}' \bigcap \mathbf{M} = CI$, i.e. $\mathbf{N}$ is irreducible in $\mathbf{M}$. Moreover, $\mathbf{N}$ is regular in $\mathbf{M}$, in other words, $\mathbf{M}$ is generated by the normalizer $\mathcal{N}_M(\mathbf{N})$.

In 1983, V. F. R. Jones asked a question in his paper "*Index for subfactors*" whether or not the converse is true [jo1, Problem 4]. Namely, if $\mathbf{N}$ is a subfactor of $\mathbf{M}$ which is regular and has trivial relative commutant, is $\mathbf{M}$ the crossed product of $\mathbf{N}$ by a group action?

In 1986, A. Ocneanu announced it at [o] for the hyperfinite II_1 factors without proof. Using the results in [pp1], S. Popa ([pp2]) gave a proof for II_1 factors and H. Kosaki ([k]) extended it to properly infinite factors. Both S. Popa and H. Kosaki's proofs essentially used C. Sutherland's vanishing cohomology result [su, Theorem 6.1].

The cohomology enters the problem as follows. We deal with unitary representatives in the cosets, which are not uniquely selected. Thus those unitaries determine a 2-cocycle action of a finite group, and so we obtain a twisted crossed product structure. In order to get a plain crossed product for V. F. R. Jones' problem, it is enough to select the representative unitaries in such a way that 2-cocycle is a coboundary. Thank to C. Sutherland [su, Theorem 6.1], it is known that a 2-cocycle is coboundary when the group is finite and the underlying algebra is a II_1 factor. Combined this with M. Pimsner and S. Popa's analysis of the relative commutants, we could give the required answer: if $\mathbf{N} \subset \mathbf{M}$ is a regular II_1 subfactors with $\mathbf{N}' \bigcap \mathbf{M} = CI$ and $[\mathbf{M} : \mathbf{N}] < \infty$, then

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M is the crossed product of N by a finite group $G = \mathcal{N}_M(\mathrm{N})/\mathcal{U}(\mathrm{N})$ acting outerly on N.

In this paper, we provide a complete proof of the converse problem without using the cohomological property. In fact, the cohomological result due to C. Sutherland is not necessary in this case. The downward basic construction of $\mathrm{N} \subset \mathbf{M}$ will work for this problem.

2. Preliminaries

2.1 Index and the basic construction for subfactors

Let **M** be a II_1 factor with the unique finite normal faithful trace τ, $\tau(I) = 1$. Denote by $L^2(\mathbf{M}, \tau)$ the completion of **M** with respect to the inner product $< x, y >= \tau(y^* x)$, for x, $y \in \mathbf{M}$.

When $\mathrm{N} \subset \mathbf{M}$ are II_1 factors with the same identity, V. F. R. Jones ([jo1]) defines the index of N in **M** as $[\mathbf{M} : \mathrm{N}] = \dim_{\mathcal{N}} L^2(\mathbf{M}, \tau)$, the Murray and von Neumann coupling constant of N in its representation by left multiplication on $L^2(\mathbf{M}, \tau)$. One of his main results in [jo1] is that $[\mathbf{M} : \mathrm{N}]$ can only take the values $\{4cos^2 \frac{\pi}{n} \mid n \geq 3\} \bigcup [4, \infty]$.

A major ingredient in the basic construction is the existence of the conditional expectation. If $\mathrm{N} \subset \mathbf{M}$ are II_1 factors, then we denote by E_N the unique trace preserving conditional expectation of **M** onto N. Also we denote by e_N the orthogonal projection of $L^2(\mathbf{M}, \tau)$ onto $L^2(\mathrm{N}, \tau)$. Thus, starting from the initial inclusion $\mathrm{N} \subset \mathbf{M}$, we can build a new algebra generated by **M** and e_N. The von Neumann algebra $\langle \mathbf{M}, e_N \rangle = \{\mathbf{M}, e_N\}''$ on $L^2(\mathbf{M}, \tau)$ is called the (upward) basic construction for $\mathrm{N} \subset \mathbf{M}$ ([g], [jo1], [pp1]).

If $[\mathbf{M} : \mathrm{N}] < \infty$, then $\langle \mathbf{M}, e_N \rangle$ is also a II_1 factor with $[\mathbf{M} : \mathrm{N}] = [\langle \mathbf{M}, e_N \rangle : \mathbf{M}]$ and $\tau(e_N) = [\mathbf{M} : \mathrm{N}]^{-1}$.

On the other hand, for a given inclusion $\mathrm{N} \subset \mathbf{M}$ with finite index, we can also build a subfactor **P** in N so that **M** is the (upward) basic construction of $\mathbf{P} \subset \mathrm{N}$ [jo1, Lemma 3.1.8], i.e., there is a unique (up to conjugacy by unitaries in N) projection $e \in \mathbf{M}$ such that $E_N(e) = [\mathbf{M} : \mathrm{N}]^{-1} I$ and $\mathbf{M} = \langle \mathrm{N}, e \rangle$. Such a construction of the subfactor is called the downward basic construction for $\mathrm{N} \subset \mathbf{M}$. More precisely, if we take $\mathbf{P} = \{e\}' \bigcap \mathrm{N}$ with $E_N(e) = [\mathbf{M} : \mathrm{N}]^{-1} I$, then **P** is the downward basic construction of $\mathrm{N} \subset \mathbf{M}$. Unlike the upward basic construction, we note that $\mathbf{P}$ is unique only up to conjugacy by unitaries in $\mathbf{N}$ [pp1, Corollary 1.8].

2.2 The relative commutant algebras

Thanks to V. F. R. Jones, A. Ocneanu, M. Pimsner and S. Popa, it turns out that the relative commutants play an important role in studying the subfactors. We first consider a finite group G acting outerly on a II_1 factor $\mathbf{M}$. When $\mathbf{N}$ is the fixed point algebra $\mathbf{M}^G$, it is well known that $\mathbf{N}$ is an irreducible regular subfactor of $\mathbf{M}$. Moreover, $\langle \mathbf{M}, e_N\rangle$ is isomorphic to the crossed product algebra $\mathbf{N} \rtimes G$, so that $\mathbf{N}'\bigcap\langle \mathbf{M}, e_N\rangle$ is isomorphic to the group algebra $\mathcal{C}[G]$.

Motivated by this phenomenon, M. Pimsner and S. Popa analyzed the minimal projections in the relative commutant $\mathbf{N}'\bigcap\langle \mathbf{M}, e_N\rangle$, as in the case of $\mathcal{C}[G]$. The following is a restatement of their result [pp1, Proposition 1.9];

PROPOSITION 2.1. *If* $[\mathbf{M} : \mathbf{N}] < \infty$ *and* $\mathbf{N}'\bigcap \mathbf{M} = \mathcal{C}I$, *then*

$$\tau(p) \geq \frac{k}{[M:N]} \quad \text{for a minimal projection } p \in \mathbf{N}'\bigcap\langle \mathbf{M}, e_N\rangle,$$

where p *belongs to a factor summand of* $\mathbf{N}'\bigcap\langle \mathbf{M}, e_N\rangle$ *isomorphic to* $M_k(\mathcal{C})$.

When $[\mathbf{M} : \mathbf{N}] < \infty$, it is known [g, Lemma 4.6.2] that $\mathbf{N}\bigcap\langle \mathbf{M}, e_N\rangle$ is a finite dimensional algebra over $\mathcal{C}$ with

$$\dim_{\mathcal{C}}(\mathbf{N}'\bigcap \langle \mathbf{M}, e_N\rangle) \leq [\mathbf{M} : \mathbf{N}]^2.$$

Due to Proposition 2.1, we have a better bound for $\dim_{\mathcal{C}}(\mathbf{N}'\bigcap\langle \mathbf{M}, e_N\rangle)$ when $\mathbf{N}$ is irreducible in $\mathbf{M}$.

COROLLARY 2.1. *For an irreducible subfactor* $\mathbf{N} \subset \mathbf{M}$ *with finite index, we have*

$$2 \leq \quad dim_{\mathcal{C}}(\mathbf{N}'\bigcap \langle \mathbf{M}, e_N\rangle) \quad \leq [\mathbf{M} : \mathbf{N}].$$

Proof. Since the Jones projection $e_N \in \mathbf{N}'\bigcap \langle \mathbf{M}, e_N\rangle$ has the property $\tau(e_N) = [\mathbf{M} : \mathbf{N}]^{-1}$, e_N is minimal and central in $\mathbf{N}'\bigcap \langle \mathbf{M}, e_N\rangle$ (cf. Proposition 2.1). So we get

$$(\mathbf{N}'\bigcap \langle \mathbf{M}, e_N\rangle)_{e_N} = \mathcal{C}e_N,$$

and hence $2 \leq \dim_{\mathcal{C}}(\mathbf{N}' \bigcap \langle \mathbf{M}, e_N \rangle)$.

Let $\mathbf{N}' \bigcap \langle \mathbf{M}, e_N \rangle = \sum_{i=1}^{k} \oplus M_{n_i}(\mathcal{C})$ be the matrix algebra decomposition with

$$(\mathbf{N}' \bigcap \langle \mathbf{M}, e_N \rangle)_{p_i} \cong M_{n_i}(\mathcal{C}),$$

where $\{p_i\}_{i=1}^{k}$ denote the minimal projections in the center of $\mathbf{N}' \bigcap \langle \mathbf{M}, e_N \rangle$. There are $\sum_{i=1}^{k} n_i$ mutually orthogonal projections, say p_i^j, in $\mathbf{N}' \bigcap \langle \mathbf{M}, e_N \rangle$ with $\sum_{i,j} p_i^j = I$, for $1 \leq j \leq n_i$. From Proposition 2.1, we have

$$\tau(p_i^j) \geq \frac{n_i}{[\mathbf{M} : \mathbf{N}]}, \text{ for all } j,$$

because $p_i^j \in (\mathbf{N}' \bigcap \langle \mathbf{M}, e_N \rangle)_{p_i} \cong M_{n_i}(\mathcal{C})$, for all j. Thus we get

$$1 = \sum_{i=1}^{k} \sum_{j=1}^{n_i} \tau(p_i^j) \geq \sum_{i=1}^{k} n_i \frac{n_i}{[\mathbf{M} : \mathbf{N}]} = \frac{1}{[\mathbf{M} : \mathbf{N}]} \sum_{i=1}^{k} n_i^2,$$

or $[\mathbf{M} : \mathbf{N}] \geq \sum_{i=1}^{k} n_i^2 = \dim_{\mathcal{C}}(\mathbf{N}' \bigcap \langle \mathbf{M}, e_N \rangle)$. □

When the depth of an irreducible subfactor $\mathbf{N} \subset \mathbf{M}$ is 2 ([g, Section 4.6] for the definition), the right hand side of the equality in Corollary 2.1 holds [s, Proposition 6]. From now on, we let $\mathbf{M}_1 = \langle \mathbf{M}, e_N \rangle$, $\mathbf{M}_2 = \langle \mathbf{M}_1, e_M \rangle$, where e_M denotes the orthogonal projection implementing the conditional expectation E_M from $\mathbf{M}_1$ onto $\mathbf{M}$.

COROLLARY 2.2. *If* $\mathbf{N}' \bigcap \mathbf{M} = \mathcal{C}I$ *and* $\mathbf{N} \subset \mathbf{M}$ *has depth 2, then*

$$\dim_{\mathcal{C}}(\mathbf{N}' \bigcap \mathbf{M}_1) = [\mathbf{M} : \mathbf{N}].$$

Proof. Let $J = (\mathbf{N}' \bigcap \mathbf{M}_1) e_M (\mathbf{N}' \bigcap \mathbf{M}_1)$ be the ideal generated by e_M in the algebra $\mathbf{N}' \bigcap \mathbf{M}_2$. Since $\mathbf{N} \subset \mathbf{M}$ has depth 2 and $\mathbf{N}' \bigcap \mathbf{M} = \mathcal{C}I$, $\mathbf{N}' \bigcap \mathbf{M}_2$ is a factor. Thus we have $\mathbf{N}' \bigcap \mathbf{M}_2 = J$. This implies that

$$\tau(I_J) = \tau(I_{N' \cap M_2}) = 1,$$

where τ denotes the normalized trace on $\mathbf{N}' \bigcap \mathbf{M}_2$.

Now let Tr be the canonical trace on $\mathbf{N}'\bigcap \mathbf{M}_2$ which takes 1 on a minimal projection. By the uniqueness of the trace on $\mathbf{N}'\bigcap \mathbf{M}_2$, there is a constant $c > 0$ such that $Tr(x) = c\,\tau(x)$ for $x \in \mathbf{N}'\bigcap \mathbf{M}_2$. Thus

$$\dim_{\mathcal{C}}(\mathbf{N}'\bigcap \mathbf{M}_1) = Tr(I_J) = c\,\tau(I_J) = c\,1 = c.$$

Since e_M is minimal in $\mathbf{N}'\bigcap \mathbf{M}_2$, $1 = Tr(e_M) = c\tau(e_M) = c\,[\mathbf{M} : \mathbf{N}]^{-1}$, or $c = [\mathbf{M} : \mathbf{N}]$. This completes the proof. $\square$

2.3 The normalizer

The normalizer of $\mathbf{N}$ in $\mathbf{M}$, denoted by $\mathcal{N}_M(\mathbf{N})$, is the set of all unitary elements in $\mathbf{M}$ that normalize $\mathbf{N}$, i.e.

$$\mathcal{N}_M(\mathbf{N}) = \{u \in \mathcal{U}(\mathbf{M}) \mid u\mathbf{N}u^* = \mathbf{N}\}.$$

Clearly, $\mathcal{N}_M(\mathbf{N})$ is a group with a normal subgroup $\mathcal{U}(\mathbf{N})$. We study the factor group $\mathcal{N}_M(\mathbf{N})/\mathcal{U}(\mathbf{N})$ (it is called the Weyl group of $\mathbf{N} \subset \mathbf{M}$), and the von Neumann algebra $\mathcal{N}_M(\mathbf{N})''$, the weakly closed $*-$subalgebra generated by $\mathcal{N}_M(\mathbf{N})$ in $\mathbf{M}$. It is clear that

$$\mathcal{N}_M(\mathbf{N})'' = \overline{span(\mathcal{N}_M(\mathbf{N}))}^{w},$$

and $\mathbf{N} \subset \mathcal{N}_M(\mathbf{N})'' \subset \mathbf{M}$.

When $[\mathbf{M} : \mathbf{N}] < \infty$, $\mathbf{M}$ is a finitely generated projective module over $\mathbf{N}$, i.e. there is a so-called Pimsner–Popa basis $\{m_i\}$ of $\mathbf{M}$ over $\mathbf{N}$ such that $E_N(m_i^* m_j) = \delta_{ij} f_i$, with f_i projections in $\mathbf{N}$ [pp1, Proposition 1.3]. Here, the projections f_i can be chosen to be identity (for all but possibly one i, which happens when $[\mathbf{M} : \mathbf{N}]$ is not an integer). A Pimsner–Popa basis $\{m_i\}$ of $\mathbf{M}$ over $\mathbf{N}$ satisfies

1. $\sum m_i m_i^* = [\mathbf{M} : \mathbf{N}]I$,
2. $\sum m_i e_N m_i^* = I$,
3. every $x \in \mathbf{M}$ can be written as $x = \sum m_i E_N(m_i^* x)$.

Thanks to a Pimsner-Popa basis of $\mathbf{M}$ over $\mathbf{N}$, we can give a characterization of the Weyl group $\mathcal{N}_M(\mathbf{N})/\mathcal{U}(\mathbf{N})$ of $\mathbf{N} \subset \mathbf{M}$ in terms of the existence of suitable projections in $\mathbf{N}\bigcap \mathbf{M}_1$. Let $G = \mathcal{N}_M(\mathbf{N})/\mathcal{U}(\mathbf{N})$ be the Weyl group of $\mathbf{N} \subset \mathbf{M}$. Choose a representative u_g of the coset $[u_g]$ in $G = \mathcal{N}_M(\mathbf{N})/\mathcal{U}(\mathbf{N})$ for each $g \in G$.

PROPOSITION 2.2. *Let* $[\mathbf{M} : \mathbf{N}] < \infty$ *and* $\mathbf{N}' \bigcap \mathbf{M} = \mathcal{C}I$. *Then we have the following:*

1. $E_N(u_g u_h^*) = 0$, *if* $g \neq h$,
2. *there is a* $1-1$ *correspondence between elements of* $G = \mathcal{N}_M(\mathbf{N})/\mathcal{U}(\mathbf{N})$ *and projections* $p \in \mathbf{N}' \bigcap \mathbf{M}_1$ *with* $E_M(p) = [\mathbf{M} : \mathbf{N}]^{-1} I$,
3. $G = \mathcal{N}_M(\mathbf{N})/\mathcal{U}(\mathbf{N})$ *is a finite group of order* $\leq [\mathbf{M} : \mathbf{N}]$.

Proof. 1. Note that $\beta = \operatorname{ad} u_g u_h^*$ determines an automorphism on N. Thus we have $u_g u_h^* x = \beta(x) u_g u_h^*$, for $x \in \mathbf{N}$. Taking E_N of both sides, we get $E_N(u_g u_h^*) x = \beta(x) E_N(u_g u_h^*)$. Since $\beta = \operatorname{ad} u_g u_h^*$ is outer if $g \neq h$, we get $E_N(u_g u_h^*) = 0$, if $g \neq h$.

2. See [pp1, Proposition 1.7].

3. Since $\mathbf{N}' \bigcap \mathbf{M}_1$ is finite dimensional, there are only finite many central projections in $\mathbf{N}' \bigcap \mathbf{M}_1$. But the projections of trace $[\mathbf{M} : \mathbf{N}]^{-1}$ are central in $\mathbf{N}' \bigcap \mathbf{M}_1$. It now follows from Property 2 that G is finite.

If $[u_g] \neq [u_h]$ in G, it follows from Property 1 that $u_g e_N u_g^*$ and $u_h e_N u_h^*$ are orthogonal if $g \neq h$. Thus $p = \sum_{g \in G} u_g e_N u_g^*$ is a projection in $\mathbf{M}_1$. Since $p \leq I$, we have

$$1 \geq \tau(p) = \sum_{g \in G} \tau(u_g e_N u_g^*) = \sum_{g \in G} \tau(u_g E_M(e_N) u_g^*) = \sum_{g \in G} [\mathbf{M} : \mathbf{N}]^{-1},$$

which implies that $[\mathbf{M} : \mathbf{N}] \geq |G|$. □

3. Main result

Note that $\{u_g \mid g \in G = \mathcal{N}_M(\mathbf{N})/\mathcal{U}(\mathbf{N})\}$ is not necessarily a group. In fact, there is a 2-cocycle $\mu : G \times G \to \mathcal{U}(\mathbf{N})$ such that

$$u_g u_h = \mu(g,h) u_{gh}, \quad \text{for each } g, h \in G.$$

Then $\operatorname{ad} u_g|_N$ determines a 2-cocycle action of G on N (for details, see [c1]).

Now we investigate the proper choices of the representative unitaries to remove the 2-cocycle μ on $G = \mathcal{N}_M(\mathbf{N})/\mathcal{U}(\mathbf{N})$.

LEMMA 3.1. *Let* $[\mathbf{M} : \mathbf{N}] < \infty$ *and* $\mathbf{N}' \bigcap \mathbf{M} = CI$. *If* $\mathbf{N}$ *is regular in* $\mathbf{M}$, *then* $\{u_g \mid g \in G = \mathcal{N}_M(\mathbf{N})/\mathcal{U}(\mathbf{N})\}$ *form a Pimsner-Popa basis of* $\mathbf{M}$ *over* $\mathbf{N}$.

Proof. Let $\mathbf{L} = \sum u_g \mathbf{N}$. It is clear that $\mathbf{L}$ is a $*-$ subalgebra of $\mathbf{M}$. Let $\{\sum u_g x_g^\alpha \mid x_g \in \mathbf{N}\}_\alpha$ be a net in $\mathbf{L}$ that converges ultraweakly to an element y in $\mathbf{M}$. For a fixed $h \in G$, we have $u_h^* \sum u_g x_g^\alpha \stackrel{\mathrm{w}}{\longrightarrow} u_h^* y$. Since E_N is ultraweakly continuous, we see that

$$E_N(u_h^* \sum u_g x_g^\alpha) \stackrel{\mathrm{w}}{\longrightarrow} E_N(u_h^* y).$$

But $E_N(u_h^* \sum u_g x_g^\alpha) = \sum E_N(u_h^* u_g) x_g^\alpha = x_h^\alpha$ (Property 1 of Proposition 2.2). Thus we have

$$x_h^\alpha \stackrel{\mathrm{w}}{\longrightarrow} E_N(u_h^* y).$$

In other words, $\sum u_g x_g^\alpha \stackrel{\mathrm{w}}{\longrightarrow} \sum u_g E_N(u_g^* y)$. Since $\sum u_g E_N(u_g^* y) \in \mathbf{L}$, we see that $\mathbf{L}$ is ultraweakly closed, i.e. $\mathbf{L} = \mathbf{M}$. This means that every element $x \in \mathbf{M}$ can be written as $x = \sum u_g x_g$, $x_g \in \mathbf{N}$. Thus we have

$$E_N(u_h^* x) = E_N(u_h^* \sum u_g x_g) = \sum E_N(u_h^* u_g) x_g = x_h,$$

and so $x = \sum u_g x_g = \sum u_g E_N(u_g^* x)$. This implies that $\{u_g\}$ form a Pimsner-Popa basis of $\mathbf{M}$ over $\mathbf{N}$. □

Lemma 3.1 indicates that $[\mathbf{M} : \mathbf{N}] = |\mathcal{N}_M(\mathbf{N})/\mathcal{U}(\mathbf{N})|$ for an irreducible regular subfactor $\mathbf{N} \subset \mathbf{M}$ with finite index. Let $[\mathbf{M} : \mathbf{N}] = n$, an integer. Denote by $\mathbf{P}$ the downward basic constructions for $\mathbf{N} \subset \mathbf{M}$ with Jones projection e_P;

$$\mathbf{P} \subset \mathbf{N} \subset \mathbf{M} \subset \mathbf{M}_1.$$

We have $\tau(e_P) = \frac{1}{n}$. Note that the central projection e_P is also minimal in the center of $\mathbf{P}' \bigcap \mathbf{M}_1$ [s, Corollary 4].

LEMMA 3.2. *Let* $[\mathbf{M} : \mathbf{N}] = n$ *and* $\mathbf{N}' \bigcap \mathbf{M} = CI$. *If* $\mathbf{N}$ *is regular in* $\mathbf{M}$, *then we have the following properties.*

1. $\mathbf{N}' \bigcap \mathbf{M}_1 \cong C^n$,
2. $\mathbf{P}' \bigcap \mathbf{M}_1 \cong M_n(C)$.

Proof. 1. By Lemma 3.1, we have $\sum u_g e_N u_g^* = I$. Since each projection $u_g e_N u_g^*$ is minimal and central in $\mathbf{N}' \bigcap \mathbf{M}_1$ (Proposition 2.1), we have $\mathbf{N}' \bigcap \mathbf{M}_1 \cong \mathcal{C}^n$.

2. Since e_P is minimal in $\mathbf{P}' \bigcap \mathbf{M}_1$, the ideal J generated by e_P is a matrix algebra. For any f, a minimal projection in $\mathbf{N}' \bigcap \mathbf{M}_1$, define $g_f = \tau(f)^{-1} f e_P f$. Since e_P is minimal in $\mathbf{P}' \bigcap \mathbf{M}_1$, $e_P f e_P = \tau(f) e_P$. Thus we have $g_f^2 = \tau(f)^{-2} f e_P f e_P f = \tau(f)^{-1} f e_P f = g_f$, i.e. g_f are mutually orthogonal projections in J. Moreover, $\tau(\sum_f g_f) = \sum_f \tau(f)^{-1} \tau(f e_P f) = \sum_f \tau(e_P) = \sum_f \frac{1}{n} = 1$. Hence $\sum_f g_f = I$. Since $I \in J$, the result follows. □

LEMMA 3.3. *Let* $[\mathbf{M} : \mathbf{N}] = n$ *and* $\mathbf{N}' \bigcap \mathbf{M} = \mathcal{C}I$. *If* $\mathbf{N}$ *is regular in* $\mathbf{M}$, *then, for each* $g \in G = \mathcal{N}_M(\mathbf{N})/\mathcal{U}(\mathbf{N})$, *there is a unique unitary* $v_g \in \mathbf{P}' \bigcap \mathbf{M}$ *such that*

1. $u_g e_N u_g^* = v_g e_N v_g^*$,
2. $v_g e_P = e_P$.

Proof. (Existence) Since $u_g e_N u_g^* \in \mathbf{N}' \bigcap \mathbf{M}_1 \subset \mathbf{P}' \bigcap \mathbf{M}_1$ is a projection of same trace as e_N, it follows from Property 2 of Lemma 3.2 that for any $g \in G$, there is an element $w_g \in \mathbf{P}' \bigcap \mathbf{M}_1$ such that $w_g e_N w_g^* = u_g e_N u_g^*$. Then, it follows from [pp1, Lemma 1.1] that there is an element $v_g \in \mathbf{P}' \bigcap \mathbf{M}$ such that $v_g e_N v_g^* = w_g e_N w_g^*$. Moreover, v_g is unitary. Indeed, $E_M(v_g e_N v_g^*) = E_M(u_g e_N u_g^*) = [\mathbf{M} : \mathbf{N}]^{-1} I$. Since $v_g \in \mathbf{M}$, we see that $v_g E_M(e_N) v_g^* = [\mathbf{M} : \mathbf{N}]^{-1} I$, or $v_g v_g^* = I$. Hence v_g is unitary and so Property 1 holds.

For Property 2, note that e_P is a minimal and central projection in $\mathbf{P}' \bigcap \mathbf{M}$. Hence $v_g e_P = c\, e_P$, for some scalar c. Replacing v_g by $c v_g$, we have the result.

(Uniqueness of v_g) Suppose that there is a unitary $w \in \mathbf{P}' \bigcap \mathbf{M}$ such that $w e_N w^* = v_g e_N v_g^*$ and $w e_P = e_P$. Since $w^* v_g$ commutes with e_N (Property 1), we have $w^* v_g \in \mathbf{M} \cap \{e_N\}' = \mathbf{N}$. Therefore, $w^* v_g \in \mathbf{P}' \bigcap \mathbf{N} = \mathcal{C}I$. i.e. there is a scalar k such that $v_g = k w$. But Property 2 implies that $e_P = v_g e_P = k w e_P = k\, e_P$, or $k = 1$. This completes the proof. □

We are now in a position to give a proof of the main theorem.

THEOREM 3.1. *Let* $\mathbf{N} \subset \mathbf{M}$ *be* II_1 *factors with* $\mathbf{N}' \bigcap \mathbf{M} = CI$. *If* $[\mathbf{M} : \mathbf{N}] = n$ *and* $\mathbf{M} = \mathcal{N}_M(\mathbf{N})''$, *then there is an outer action of* G *on* $\mathbf{N}$ *such that*

$$\mathbf{M} = \mathcal{N}_M(\mathbf{N})'' = \mathbf{N} \rtimes G,$$

where $G = \mathcal{N}_M(\mathbf{N})/\mathcal{U}(\mathbf{N})$ *is a finite group of order* n.

Proof. It follows from the uniqueness pointed in Lemma 3.3 that $\{v_g \,|\, g \in G\}$ form a group. Thus, $\mathrm{ad}\, v_g|_N$ determines an action of G on $\mathbf{N}$. Moreover, α_g is outer if $g \neq e$. Indeed, if it is inner, then there is a unitary $w_g \in \mathbf{N}$ such that $\alpha = \mathrm{ad} w_g = \mathrm{ad} v_g$, for all $g \neq e$. Thus $v_g^* w_g \in \mathbf{N}' \bigcap \mathbf{M} = CI$, and so there is a scalar k_g such that $w_g = k_g v_g \in \mathbf{N}$, a contradiction. Hence $\mathbf{N} \rtimes G$ is also a II_1 factor. Since $\mathbf{N} \rtimes G$ is generated by $\mathbf{N}$ and $\{v_g | g \in G\}$, we see that $\mathbf{N} \rtimes G = \mathcal{N}_M(\mathbf{N})''$. □

4. An application to depth 2 subfactors

When $\mathbf{M} = \mathbf{N} \rtimes G$ is the crossed product by an outer action of a finite group G on $\mathbf{N}$, $\mathbf{N} \subset \mathbf{M}$ has depth 2 [g, Section 4.7]. In particular, $\mathbf{N}' \bigcap \mathbf{M} = CI$, and the relative commutant $\mathbf{N}' \bigcap \mathbf{M}_1$ is a $|G|$ dimensional abelian algebra.

In 1986, A. Ocneanu announced that the converse is true for the hyperfinite II_1 factors ([o]): if $\mathbf{N} \subset \mathbf{M}$ are hyperfinite II_1 factors with $[\mathbf{M} : \mathbf{N}] = n$, $\mathbf{N}' \bigcap \mathbf{M} = CI$, $\mathbf{N}' \bigcap \mathbf{M}_1$ is abelian, and $\mathbf{N} \subset \mathbf{M}$ has depth 2, then $\mathbf{M}$ is the crossed product algebra $\mathbf{N} \rtimes G$ by an outer action of G with $|G| = n$. Due to Corollary 2.2, his conditions can be replaced by the following; $\mathbf{N}' \bigcap \mathbf{M} = CI$. $\mathbf{N}' \bigcap \mathbf{M}_1$ is a $[\mathbf{M} : \mathbf{N}]$ dimensional abelian algebra.

We relate Theorem 3.1 to a characterization of crossed product algebras for II_1 factors. Note that the order of the Weyl group G is determined by the number of mutually orthogonal projections p in $\mathbf{N}' \bigcap \mathbf{M}_1$ with $E_M(p) = [\mathbf{M} : \mathbf{N}]^{-1} I$ (Proposition 2.2). In the case of $\mathbf{N}' \bigcap \mathbf{M} = CI$, $E_M(p)$ belongs to $\mathbf{N}' \bigcap \mathbf{M}$, so it is simply $\tau(p)I$. Consequently, to determine the size of G in the crossed product algebra $\mathcal{N}_M(\mathbf{N})'' = \mathbf{N} \rtimes G$, it is sufficient to find projections in $\mathbf{N}' \bigcap \mathbf{M}_1$, each of trace $[\mathbf{M} : \mathbf{N}]^{-1}$. This method is particularly useful in the case when $\mathbf{N}' \bigcap \mathbf{M}_1$ contains $[\mathbf{M} : \mathbf{N}]$ such projections. The condition of

[M : N] dimensional abelian algebra guarantees the existence of such projections. We have the following characterization of crossed product algebras [pp2, Corollary 1.1.6].

THEOREM 4.1. *Let* $\mathrm{N} \subset \mathrm{M}$ *be* II_1 *factors with* $[\mathrm{M} : \mathrm{N}] = n$ *and* $\mathrm{N}' \bigcap \mathrm{M} = \mathcal{C}I$. *If* $\mathrm{N}' \bigcap \mathrm{M}_1$ *is an n-dimensional abelian algebra, then* M *is the crossed product algebra* $\mathrm{N} \rtimes G$, *where* G *is a group of order* n *acting outerly on* N.

Proof. Since $\mathrm{N}' \bigcap \mathrm{M}_1 \cong \mathcal{C}^n$, there exist n mutually orthogonal minimal and central projections, say p_i $(1 \leq i \leq n)$, each of trace $\frac{1}{n}$. Since $E_M(p_i) \in \mathrm{N}' \bigcap \mathrm{M} = \mathcal{C}I$ for any projection $p_i \in \mathrm{N}' \bigcap \mathrm{M}_1$, $E_M(p_i) = \tau(p_i)I = \frac{1}{n}I = [\mathrm{M} : \mathrm{N}]^{-1}I$, for all i. Thus we have $|G| = |\mathcal{N}_M(\mathrm{N})/\mathcal{U}(\mathrm{N})| = n$ (Proposition 2.2). It follows from Theorem 3.1 that there is an outer action of G on N such that $\mathcal{N}_M(\mathrm{N})'' = \mathrm{N} \rtimes G$. The multiplicative property of Jones index [jo1, Proposition 2.1.8] for the tower

$$\mathrm{N} \subset \mathcal{N}_M(\mathrm{N})'' = \mathrm{N} \rtimes G \subset \mathrm{M}$$

gives that

$$n = [\mathrm{M} : \mathrm{N}] = [\mathrm{M} : \mathrm{N} \rtimes G][\mathrm{N} \rtimes G : \mathrm{N}] = [\mathrm{M} : \mathrm{N} \rtimes G]\, n.$$

Thus we have $[\mathrm{M} : \mathrm{N} \rtimes G] = 1$, and so $\mathrm{M} = \mathrm{N} \rtimes G$ [jo1, Proposition 2.1.8]. □

The duality of the crossed product algebras and the fixed point algebras gives the following corollary as an immediate consequence of Theorem 4.1.

COROLLARY 4.1. *If* $\mathrm{N} \subset \mathrm{M}$ *are* II_1 *factors with* $\mathrm{N}' \bigcap \mathrm{M} = \mathcal{C}I$ *and* $\mathrm{M}' \bigcap \mathrm{M}_2$ *is a* $[\mathrm{M} : \mathrm{N}] = n$ *dimensional abelian algebra, then there is an outer action of a finite group* G *with* $|G| = n$ *such that* N *is the fixed point algebra* M^G.

REMARK 4.1. These characterizations of crossed product algebras can be extended to properly infinite factors due to H. Kosaki ([k]).

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