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MINIMUM PERMANENT ON THE POLYTOPES DETERMINED BY A VECTOR MAJORIZATION*

GI-SANG CHEON

1. Introduction

Let Ω_n denote the set of all $n \times n$ doubly stochastic matrices. Then it is well known that Ω_n forms a convex polytope of dimension $(n-1)^2$ with $n!$ extreme points in the n^2-dimensional Euclidean space.

For an $n \times n$ (0,1)-matrix $U = [u_{ij}]$, let

$$\mathcal{F}(U) = \{X = [x_{ij}] \in \Omega_n | x_{ij} = 0 \text{ whenever } u_{ij} = 0\}.$$

Then $\mathcal{F}(U)$ is a face of the polytope Ω_n.

One of the most interesting problems concerning the polytope $\mathcal{F}(U)$ is that of determining the minimum value of the permanent function and the set of all minimizing matrices on it.

For integers k, n with $1 \leq k \leq n$, let $V_{k,n}$ denote the set of all $n \times 1$ (0,1)-matrices whose entries have sum k, and let $\mathbf{R}^n$ denote the set of all real $n \times 1$ matrices. For $\mathbf{x}$, $\mathbf{y} \in \mathbf{R}^n$, $\mathbf{y}$ is said to be *majorized* by $\mathbf{x}$, written as $\mathbf{y} \prec \mathbf{x}$, if

$$\max\{\mathbf{v}^{\mathrm{T}}\mathbf{y} | \mathbf{v} \in V_{k,n}\} \leq \max\{\mathbf{v}^{\mathrm{T}}\mathbf{x} | \mathbf{v} \in V_{k,n}\} \tag{1.1}$$

for all $k = 1, \cdots, n$ and the equality holds in (1.1) when $k = n$.

It is well known that $\mathbf{y} \prec \mathbf{x}$ for $\mathbf{x}$, $\mathbf{y} \in \mathbf{R}^n$ if and only if there exists $S \in \Omega_n$ with $\mathbf{y} = S\mathbf{x}$. Let

$$\Omega_n(\mathbf{y} \prec \mathbf{x}) = \{S \in \Omega_n | \mathbf{y} = S\mathbf{x}\}. \tag{1.2}$$

Then $\Omega_n(\mathbf{y} \prec \mathbf{x})$ forms a convex subpolytope of Ω_n. We call it the *polytope of the majorization* $\mathbf{y} \prec \mathbf{x}$.

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In [1], R.A. Brualdi determined the dimensions of majorization polytopes and the support matrix of the majorization. But as remarked by Marshall and Olkin [5], very little is known about this polytope.

Let $\mathcal{F}(U)$ be the face determined by the support matrix U of the majorization. Then $\mathcal{F}(U)$ is the largest face of Ω_n whose interior has a nonempty intersection with the majorization polytope $\Omega_n(\mathbf{y} \prec \mathbf{x})$.

The purpose of this paper is to determine the minimum permanent and the set of all minimizing matrices on the polytope $\mathcal{F}(U)$.

2. The support matrix of the majorization

Throughout this paper, let $\mathbf{x} = (x_1, \cdots, x_n)^{\mathrm{T}}$ and $\mathbf{y} = (y_1, \cdots, y_n)^{\mathrm{T}}$ be in $\mathbf{R}^n$ with $\mathbf{y} \prec \mathbf{x}$, and without loss of generality, we may assume that $x_1 \geq \cdots \geq x_n$ and $y_1 \geq \cdots \geq y_n$.

For integers k, n with $1 \leq k \leq n$, we say that $\mathbf{y} \prec \mathbf{x}$ has a *coincidence* at k if $y_1 + \cdots + y_k = x_1 + \cdots + x_k$, and is *$k$-decomposable* if it has a coincidence at k $(< n)$ and $x_k > x_{k+1}$.

Suppose that $\mathbf{y} \prec \mathbf{x}$ has a coincidence at k $(< n)$, but is not k-decomposable. Then there exist integers j_i, k_i, k_i' and l_i, $(i = 1, \cdots, p)$, such that

(2.1)
$$\begin{cases} (i) & \text{the only coincidences of } \mathbf{y} \prec \mathbf{x} \text{ occur at } k_i,\ k_i + 1, \cdots, k_i', \\ (ii) & k_{i-1}' < l_{i-1} < j_i \leq k_i \leq k_i', \\ (iii) & x_{j_i - 1} > x_{j_i} = \cdots = x_{k_i} = \cdots = x_{k_i'} = \cdots = x_{l_i} > x_{l_i+1}. \end{cases}$$

Note that $k_p' = l_p = n$, and we shall also use $k_0' = l_0 = 0$ and $j_0 = 1$.

For $A = [a_{ij}] \in \Omega_n$, we define the *support matrix* of A to be the $n \times n$ (0,1)-matrix $S_A = [s_{ij}]$ by

$$s_{ij} = \begin{cases} 1 & \text{if } a_{ij} > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathbf{x}$, $\mathbf{y} \in \mathbf{R}^n$ with $\mathbf{y} \prec \mathbf{x}$. We define the *support matrix of the majorization* $\mathbf{y} \prec \mathbf{x}$ to be the $n \times n$ (0,1)-matrix $U = [u_{ij}]$ having the following two properties:

(i) $A \in \Omega_n(\mathbf{y} \prec \mathbf{x})$ implies $A \leq U$

(ii) there is a matrix $B = [b_{ij}] \in \Omega_n(\mathbf{y} \prec \mathbf{x})$ such that $S_B = U$.

THEOREM 2.1. (R.A. Brualdi [1]) *Let* $\mathbf{x},\ \mathbf{y} \in \mathbf{R}^n$ *and suppose* $\mathbf{y} \prec \mathbf{x}$ *is not* k*-decomposable for each* $k = 1, \cdots, n-1$. *Then the support matrix of* $\mathbf{y} \prec \mathbf{x}$ *is the* $n \times n$ $(0,1)$*-matrix* $U = [u_{rs}]$ *such that for* $1 \le r, s \le n$, $u_{rs} = 1$ *if and only if for some* $i = 1, \cdots, p$,

$$k'_{i-1} + 1 \le r \le k_i,\ j_{i-1} \le s \le l_i \ \text{ or } \ k_i + 1 \le r \le k'_i,\ j_i \le s \le l_i.$$

In the following we represent the support matrix of the majorization $\mathbf{y} \prec \mathbf{x}$ in a different manner.

With j_i, k_i, k'_i and l_i, $(i = 1, \cdots, p)$, defined by (2.1) we define frames σ and σ' of U by

$$\begin{aligned} &\sigma = (k_1, \cdots, k_p, n : j_1 - 1, \cdots, j_p - 1, n) \qquad \text{and} \\ &\sigma' = (k'_1, \cdots, k'_p : l_1, \cdots, l_p). \end{aligned} \tag{2.2}$$

In particular, if $k_p = k'_p$ then we denote σ by

$$\sigma = (k_1, \cdots, k_p : j_1 - 1, \cdots, j_{p-1} - 1, n). \tag{2.3}$$

Throughout this paper, $K_{p \times q}$ will denote the $p \times q$ matrix all of whose entries are 1 and K the matrix of 1's of suitable sizes.

For σ and σ' defined by (2.2), let $U_{\sigma\sigma'} = [U_{ij}]$ be $n \times n$ (0,1)-matrix with

$$U_{ij} = \begin{cases} K_{g(t) \times h(t)} & \text{if } 2t - 1 \le i \le j \le 2t \quad \text{and} \\ & i = 2t - 1,\ j = 2t - 2,\ (t = 1, \cdots, p) \\ 0 & \text{otherwise} \end{cases}$$

where, for some $t = 1, \cdots, p$,

$$g(t) = \begin{cases} k_t - k'_{t-1} & \text{if } i = 2t - 1 \\ k'_t - k_t & \text{if } i = 2t \end{cases} \quad \text{and}$$

$$h(t) = \begin{cases} j_t - l_{t-1} - 1 & \text{if } j = 2t - 1 \\ l_t - j_t + 1 & \text{if } j = 2t. \end{cases}$$

Note that there is no $U_{2t\,\cdot}$ for t with $k'_t = k_t$ and $U_{\cdot\,(2t-1)}$ for t with $j_t = l_{t-1} + 1$. Thus those submatrices do not appear in the matrix $U_{\sigma\sigma'}$.

Then the (0,1)-matrix $U_{\sigma\sigma'}$ is the support matrix of $\mathbf{y} \prec \mathbf{x}$, and $U_{\sigma\sigma'}$ has the following block representation

(2.4) $U_{\sigma\sigma'} =$

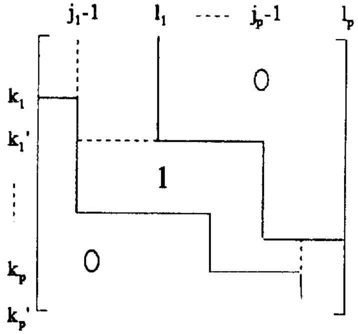


Figure 1

EXAMPLE 2.1.

Let

(2.5)
$$\mathbf{y} = (8,7,7,6,5,5,4,4,2,2)^T \text{ and } \mathbf{x} = (9,7,6,6,6,5,4,3,3,1)^T.$$

Then $\mathbf{y} \prec \mathbf{x}$ but is not k-decomposable. It has 4 coincidences, namely at $k_1 = 3,\ 4 = k_1',\ k_2 = 8 = k_2',\ k_3 = 10 = k_3'$. Moreover, $j_1 = 3,\ l_1 = 5,\ j_2 = 8,\ l_2 = 9,\ j_3 = l_3 = 10$.

From (2.3) and (2.2), we get the frames

(2.6) $$\sigma = (3,8,10 \ : \ 2,7,10), \quad \sigma' = (4,8,10 \ : \ 5,9,10)$$

and the support matrix $U_{\sigma\sigma'}$ of $\mathbf{y} \prec \mathbf{x}$ is

(2.7) $$U_{\sigma\sigma'} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

3. The barycenter of the polytope $\mathcal{F}(U_{\sigma\sigma'})$ and its permanent

An $n \times n$ matrix is called *partly decomposable* if it containes a $k \times (n-k)$ zero submatrix for some k, $1 \le k \le n-1$, and an n-square matrix which is not partly decomposable is called *fully indecomposable.* A matrix $A \in \mathcal{F}(U)$ is called a *minimizing matrix* on $\mathcal{F}(U)$ if

$$\text{per } A = \min\{ \text{ per } X | X \in \mathcal{F}(U)\}.$$

For a matrix A, $A(i_1, \cdots, i_s|j_1, \cdots, j_t)$ will denote the $(n-s)\times(n-t)$ matrix obtained from A by deleting the rows $i_1, \cdots, i_s$ and the columns $j_1, \cdots, j_t$, and $A[i_1, \cdots, i_s|j_1, \cdots, j_t]$ the $s \times t$ submatrix of A complementary to $A(i_1, \cdots, i_s|j_1, \cdots, j_t)$. In particular, if we delete the rows $i_1, \cdots, i_s$ only, the resulting matrix is denoted by $A(i_1, \cdots, i_s| \cdot)$. Similarly, the matrices $A(\cdot|j_1, \cdots, j_t)$, $A[i_1, \cdots, i_s| \cdot]$ and $A[\cdot|j_1, \cdots, j_t]$ are defined.

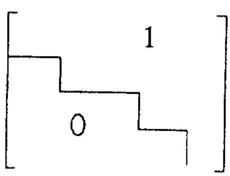
For $\mathbf{x}$, $\mathbf{y} \in \mathbf{R}^n$, suppose that $\mathbf{y} \prec \mathbf{x}$ has a coincidence at k $(< n)$ but is not k-decomposable. With frames $\sigma = (k_1, \cdots, k_p, n \ : \ j_1 - 1, \cdots, j_p - 1, n)$ and $\sigma' = (k'_1, \cdots, k'_p \ : \ l_1, \cdots, l_p)$ defined by (2.2), we define some quantities α_i's, β_i's, γ_i's and δ_i's, $i = 1, \cdots, p$, by

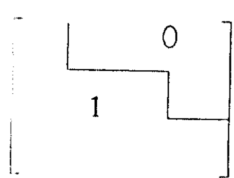
$$(3.1) \quad \alpha_i = k_i - j_i + 1, \ \beta_i = l_i - k'_i, \ \gamma_i = l_i - j_i + 1, \text{ and } \delta_i = k_i - k'_{i-1},$$

and let $U_{\sigma\sigma'}$ be the support matrix of the majorization $\mathbf{y} \prec \mathbf{x}$. Then by Theorem 2.5 of [2], we get

$$\begin{aligned}\dim\mathcal{F}(U_{\sigma\sigma'}) &= (n-1)^2 - \sum_{i=1}^{p}(j_i - j_{i-1})(n-k_i) - \sum_{i=1}^{p-1}(k'_i - k'_{i-1})(n-l_i) \\ &= \dim\mathcal{F}(U_\sigma) + \dim\mathcal{F}(U_{\sigma'}) - \dim\Omega_n\end{aligned}$$

where

$U_\sigma =$ 

and $U_{\sigma'} =$ 

Note that S.G. Hwang [4] determined the set of all minimizing matrices and its minimum permanent on $\mathcal{F}(U_\sigma)$ or $\mathcal{F}(U_{\sigma'})$.

It readily follows by induction that the number of vertices of $\mathcal{F}(U_{\sigma\sigma'})$, per $U_{\sigma\sigma'}$, is

$$\prod_{i=1}^{p} \frac{\gamma_i!\delta_i!}{\alpha_i!\beta_i!} \tag{3.2}$$

where α_i, β_i, γ_i and δ_i are defined by (3.1).

Now, we consider the barycenter $B(U_{\sigma\sigma'})$ of $\mathcal{F}(U_{\sigma\sigma'})$ given by

$$B(U_{\sigma\sigma'}) = \frac{1}{\text{per } U_{\sigma\sigma'}} \sum_{P \le U_{\sigma\sigma'}} P \tag{3.3}$$

where the summation extends over the set of all permutation matrices P with $P \le U_{\sigma\sigma'}$.

Finally, we define an $n \times n$ matrix $M_{\sigma\sigma'}$ with the same block representation as $U_{\sigma\sigma'}$ in (2.4) by

$$M_{\sigma\sigma'} = \begin{bmatrix} M_{11} & \cdots & M_{1\,2p} \\ \vdots & & \vdots \\ M_{2p\,1} & \cdots & M_{2p\,2p} \end{bmatrix} \tag{3.4}$$

where, for each $t = 1, \cdots, p$,

$$M_{ij} = \begin{cases} r_t K & \text{if } i = j = 2t-1, \\ s_t K & \text{if } i = 2t-1,\ j = 2t, \\ t_t K & \text{if } i = j = 2t, \\ q_t K & \text{if } i = 2t-1,\ j = 2t-2. \\ 0 & \text{otherwise} \end{cases} \tag{3.5}$$

where

$$r_t = \frac{1}{\delta_t},\quad s_t = \frac{\alpha_t}{\gamma_t\delta_t}\quad t_t = \frac{1}{\gamma_t} \text{ and } q_t = \frac{\beta_{t-1}}{\gamma_{t-1}\delta_t}. \tag{3.6}$$

THEOREM 3.1. *Let $U_{\sigma\sigma'}$ be the support matrix of* $\mathbf{y} \prec \mathbf{x}$ *with* σ *and* σ' *given by* (2.2). *Then* $B(U_{\sigma\sigma'}) = M_{\sigma\sigma'}$.

Proof. Let $B(U_{\sigma\sigma'}) = [b_{rs}]_{n\times n}$ be the barycenter of $\mathcal{F}(U_{\sigma\sigma'})$. Then since

$$b_{rs} = \frac{\text{per } U_{\sigma\sigma'}(r|s)}{\text{per } U_{\sigma\sigma'}}, \quad (r, s = 1, \cdots, n)$$

$B(U_{\sigma\sigma'}) = [B_{ij}]_{n\times n}$ has the same zero-one pattern as $U_{\sigma\sigma'}$ and the (i,j)-block B_{ij} has the same entries. Thus we have

$$(3.7) \qquad b_{rs}\text{per } U_{\sigma\sigma'} = \begin{cases} \text{per } U_{\sigma\sigma'}(k_t|j_t - 1) & \text{if } b_{rs} \in B_{2t-1\,2t-1}, \\ \text{per } U_{\sigma\sigma'}(k_t|l_t) & \text{if } b_{rs} \in B_{2t-1\,2t}, \\ \text{per } U_{\sigma\sigma'}(k'_t|l_t) & \text{if } b_{rs} \in B_{2t\,2t}, \\ \text{per } U_{\sigma\sigma'}(k_t|l_{t-1}) & \text{if } b_{rs} \in B_{2t-1\,2t-2} \end{cases}$$

for each $t = 1, \cdots, p$.

If we notice that $k'_{t-1} < l_{t-1} < j_t \le k_t \le k'_t < l_t$ for each $t = 1, \cdots, p$, then from (3.1), (3.2) and (3.7) it follows that $B(U_{\sigma\sigma'}) = M_{\sigma\sigma'}$.

For α_i, β_i, γ_i and δ_i $(i = 1, \cdots, p)$ given by (3.1), we define some ordered pair (π_t, θ_t) by

$$(3.8) \qquad (\pi_t, \theta_t) = \begin{cases} (\delta_i - \alpha_i - \beta_{i-1}, \delta_i - \beta_{i-1}) & \text{if } t = 4i-3, \\ (\alpha_i, \gamma_i) & \text{if } t = 4i-2, \\ (\gamma_i - \alpha_i - \beta_i, \gamma_i - \alpha_i) & \text{if } t = 4i-1, \\ (\beta_{i-1}, \delta_i) & \text{if } t = 4i-4. \end{cases}$$

Notice that, for each $t = 1, \cdots, 2p-1$,

$$(3.9) \qquad \begin{cases} \theta_{2t-1} - \pi_{2t-1} = \pi_{2t} \\ \theta_{2t} - \pi_{2t} = \theta_{2t+1}. \end{cases}$$

THEOREM 3.2. *Let* α_i, β_i, γ_i *and* δ_i *be as defined in* (3.1). *Then*

$$(3.10) \qquad \text{per } M_{\sigma\sigma'} = \prod_{i=1}^{p} \frac{\gamma_i!\delta_i!}{\alpha_i!\beta_i!}\frac{\alpha_i^{\alpha_i}\beta_i^{\beta_i}}{\gamma_i^{\gamma_i}\delta_i^{\delta_i}}.$$

Proof. First, we let

$$\phi_{\sigma\sigma'} = \prod_{i=1}^{p} \Delta(i), \qquad \Delta(i) = \frac{\gamma_i!\delta_i!}{\alpha_i!\beta_i!}\frac{\alpha_i^{\alpha_i}\beta_i^{\beta_i}}{\gamma_i^{\gamma_i}\delta_i^{\delta_i}},$$

and for an $n \times n$ matrix A, let

$$\text{per } A(1, \cdots, \hat{\pi}_i | 1, \cdots \hat{\pi}_i) = P_{\hat{\pi}_i}(A)$$

where $\hat{\pi}_i = \sum_{j=1}^{i} \pi_j$.

Then we get the following recursion formula for per $M_{\sigma\sigma'}$ using the Laplace expansion for permanent:

$$(3.11)\quad P_{\hat{\pi}_{t-1}}(M_{\sigma\sigma'}) = \frac{\theta_t!}{(\theta_t - \pi_t)!} x_{f(t)}^{\pi_t} P_{\hat{\pi}_t}(M_{\sigma\sigma'}), \quad (t = 1, \cdots, 4p-1)$$

where, for each $i = 1, \cdots, p$,

$$(3.12)\qquad x_{f(t)} = \begin{cases} r_i & \text{if } t = 4i-3, \\ s_i & \text{if } t = 4i-2, \\ t_i & \text{if } t = 4i-1, \\ q_i & \text{if } t = 4i-4 \end{cases}$$

where r_i, s_i, t_i and q_i are defined by (3.6). Note that

$$P_{\hat{\pi}_0}(M_{\sigma\sigma'}) = \text{per } M_{\sigma\sigma'} \quad \text{and} \quad P_{\hat{\pi}_{4p-1}}(M_{\sigma\sigma'}) = 1.$$

Thus from (3.11) we have

$$(3.13)\qquad \text{per } M_{\sigma\sigma'} = \prod_{t=1}^{4p-1} \frac{\theta_t!}{(\theta_t - \pi_t)!} \prod_{t=1}^{4p-1} x_{f(t)}^{\pi_t}$$

On the other hand, from (3.8) and (3.9), it is easily seen that

$$(3.14)\qquad \prod_{t=1}^{4p-1} \frac{\theta_t!}{(\theta_t - \pi_t)!} = \prod_{i=1}^{p} \frac{\gamma_i!\delta_i!}{\alpha_i!\beta_i!}$$

and also from (3.8), (3.9) and (3.12), we get

$$\prod_{t=1}^{4p-1} x_{f(t)}^{\pi_t} = \prod_{i=1}^{p} \frac{\alpha_i^{\alpha_i}\beta_i^{\beta_i}}{\gamma_i^{\gamma_i}\delta_i^{\delta_i}}. \tag{3.15}$$

(We used the convension $0^0 = 1$).

Therefore from (3.13), (3.14) and (3.15), we have

$$\text{per } M_{\sigma\sigma'} = \phi_{\sigma\sigma'}$$

which completes the proof.

For the Example 2.1, we get $\alpha_1 = 1$, $\alpha_2 = 1$, $\alpha_3 = 1$; $\beta_1 = 1$, $\beta_2 = 1, \beta_3 = 0$; $\gamma_1 = 3$, $\gamma_2 = 2$, $\gamma_3 = 1$; $\delta_1 = 3$, $\delta_2 = 4$, $\delta_3 = 2$ and $r_1 = \frac{1}{3}$, $r_2 = \frac{1}{4}$; $s_1 = \frac{1}{9}$, $s_2 = \frac{1}{8}$, $s_3 = \frac{1}{2}$; $t_1 = \frac{1}{3}$; $q_2 = \frac{1}{12}$, $q_3 = \frac{1}{4}$. (Note that q_1, t_2, t_3 and r_3 do not appear.)

Thus we have

$$M_{\sigma\sigma'} = \begin{bmatrix} 1/3 & 1/3 & 1/9 & 1/9 & 1/9 & 0 & 0 & 0 & 0 & 0 \\ 1/3 & 1/3 & 1/9 & 1/9 & 1/9 & 0 & 0 & 0 & 0 & 0 \\ 1/3 & 1/3 & 1/9 & 1/9 & 1/9 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/3 & 1/3 & 1/3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/12 & 1/12 & 1/12 & 1/4 & 1/4 & 1/8 & 1/8 & 0 \\ 0 & 0 & 1/12 & 1/12 & 1/12 & 1/4 & 1/4 & 1/8 & 1/8 & 0 \\ 0 & 0 & 1/12 & 1/12 & 1/12 & 1/4 & 1/4 & 1/8 & 1/8 & 0 \\ 0 & 0 & 1/12 & 1/12 & 1/12 & 1/4 & 1/4 & 1/8 & 1/8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/4 & 1/4 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/4 & 1/4 & 1/2 \end{bmatrix}$$

and

$$\text{per } M_{\sigma\sigma'} = \frac{3!3!2!4!2!}{3^3 3^3 2^2 4^4 2^2} = \frac{1}{864}.$$

4. The minimum permanent and the minimizing matrices

The following Lemma is due to Foregger [3].

LEMMA 4.1. *Let $U = [u_{ij}]$ be an $n \times n$ fully indecomposable $(0,1)$-matrix, and let $A = [a_{ij}]$ be a minimizing matrix on the face $\mathcal{F}(U)$. Then A is fully indecomposable, and more, for (i,j) such that $u_{ij} = 1$*

$$\text{per } A(i|j)\begin{cases} = \text{per } A & \text{if } a_{ij} > 0 \\ \geq \text{per } A & \text{if } a_{ij} = 0. \end{cases}$$

And we need the following Lemma due to Minc [6].

LEMMA 4.2. *Let $A = [a_{ij}]$ be a minimizing matrix on $\mathcal{F}(U)$ where $U = [u_1, \cdots, u_n]$ is an $n \times n$ $(0,1)$-matrix. If, for some $k \leq n$, $d_1 = \cdots = d_k$, then for any $p \leq k$, $A(J_p \oplus I_{n-p}) \in \mathcal{F}(U)$ and per $A(J_p \oplus I_{n-p}) =$ per A. From now on in the sequal let J_p denote the p-square matrix all of whose entries are $1/p$ and I the square identity matrix, i.e., the matrix obtained from A by replacing each of its first p columns by their average remains a minimizing matrix on $\mathcal{F}(U)$. A similar statement holds for rows.*

Let $U_{\sigma\sigma'}$ be the support matrix of the majorization $\mathbf{y} \prec \mathbf{x}$ with σ and σ' given by (2.2), and let $M_{\sigma\sigma'}$ be the matrix defined by (3.4). Since $U_{\sigma\sigma'}$ has a block representation, by Lemma 4.2, it follows that $M_{\sigma\sigma'}$ is a minimizing matrix on $\mathcal{F}(U_{\sigma\sigma'})$. Hence, by Theorem 3.2, we get the following.

THEOREM 4.1. *Let $U_{\sigma\sigma'}$ be the support matrix of the majorization $\mathbf{y} \prec \mathbf{x}$ with σ and σ' given by (2.2). Then*

$$\min_{X \in \mathcal{F}(U_{\sigma\sigma'})} \text{per } X = \phi_{\sigma\sigma'}. \tag{4.1}$$

With j_i, k_i, k'_i and l_i defined in (2.1), let

$$C_{\sigma\sigma'} = \{t | k_t = j_t, \quad t = 1, \cdots, p\} \tag{4.2}$$

and

$$\widehat{C}_{\sigma\sigma'} = \{s | k'_s = l_s - 1, \quad s = 1, \cdots, p-1\}. \tag{4.3}$$

Suppose that $C_{\sigma\sigma'} \neq \emptyset$ or $\widehat{C}_{\sigma\sigma'} \neq \emptyset$. Let $C_{\sigma\sigma'} = \{t_1, \cdots, t_k\}$ and $\widehat{C}_{\sigma\sigma'} = \{s_1, \cdots, s_l\}$, and let $m_1, \cdots, m_{k+l}$ be positive integers obtained from $2t_1-1,\cdots,2t_k-1,2s_1,\cdots,2s_l$ by rearranging in increasing order. Then we get a sequence

$$(4.4)\qquad m_1,\cdots,m_{k+l}.$$

Let $\mathcal{I}$ be the set of all indices of the sequence with nonconsecutive index in (4.4) and let t_o and t_e be an odd and an even number in $\mathcal{I}$ respectively. Notice that

$$\begin{cases} t\in C_{\sigma\sigma'} & \text{if } t_o=2t-1,\ t=1,\cdots,p \\ s\in \widehat{C}_{\sigma\sigma'} & \text{if } t_e=2s,\ s=1,\cdots,p-1. \end{cases}$$

Finally, we define a matrix $M^*_{\sigma\sigma'}=[M^*_{ij}]$ on $\mathcal{F}(U_{\sigma\sigma'})$ obtained from $M_{\sigma\sigma'}=[M_{ij}]$ by

$$(4.5)\qquad M^*_{ij}=\begin{cases} S_i & \text{if } i=t_o,\ j=t_o+1 \text{ for } t_o\in\mathcal{I} \\ S_j & \text{if } i=t_e+1,\ j=t_e \text{ for } t_e\in\mathcal{I} \\ M_{ij} & \text{otherwise.} \end{cases}$$

where the entries of S_i and S_j can be chosen freely as long as $M^*_{\sigma\sigma'}$ remains doubly stochastic.

Let $\tau(S)$ be the sum of all the entries in the matrix S. Then $\tau(S_{2t-1})=k_t-j_t+1$ for $t=1,\cdots,p$ and $\tau(S_{2s})=l_s-k'_s$ for $s=1,\cdots,p-1$. Thus we have the following.

LEMMA 4.3. *Let $C_{\sigma\sigma'}$ and $\widehat{C}_{\sigma\sigma'}$ be defined in (4.2) and (4.3). Then $\tau(S_{2t-1})=1$ for $t\in C_{\sigma\sigma'}$, and $\tau(S_{2s})=1$ for $s\in\widehat{C}_{\sigma\sigma'}$.*

LEMMA 4.4. *For each t and s such that $2t-1=t_o$ and $2s=t_e$ for $t_o,\ t_e\in\mathcal{I}$,*

$$P_{\pi_{h(k)}}(M^*_{\sigma\sigma'})=\left\{\prod_{i=h(k)+1}^{\pi_{h(k)+3}}\Delta(i)\right\}P_{\pi_{h(k)+3}}(M_{\sigma\sigma'})$$

where

$$(x_k,y_k)=\begin{cases} (q_t,r_t) & \text{if } h(t)=4t-5 \text{ for } k=t,\ t=2,\cdots,p \\ (s_s,t_s) & \text{if } h(s)=4s-3 \text{ for } k=s,\ s=1,\cdots,p-1. \end{cases}$$

Proof. First, let $h(t) = 4t - 5$ for $k = t$, $t = 2, \cdots, p$. For a matrix $M^*_{\sigma\sigma'} = [M^*_{ij}]$ with $t \in C_{\sigma\sigma'}$ we get

$$M^*_{ij} = \begin{cases} S_i & \text{if } i = 2t-1, \ j = 2t \\ M_{ij} & \text{otherwise.} \end{cases}$$

Thus if we notice that $\pi_{4t-2} = 1$ and $s_t = \frac{1}{\theta_{4t-2}\theta_{4t-4}}$, then, by (3.8), (3.9), (3.11) and Lemma 4.3, we get

$$\begin{aligned} P_{\hat{\pi}_{h(t)}}(M^*_{\sigma\sigma'}) &= (\theta_{h(t)+1} - 1)! q_t^{\pi_{h(t)+1}} r_t^{\pi_{h(t)+2}} \tau(S_{2t-1}) P_{\hat{\pi}_{h(t)+3}}(M^*_{\sigma\sigma'}) \\ &= \left\{ \prod_{i=h(t)+1}^{\pi_{h(t)+3}} \Delta(i) \right\} P_{\hat{\pi}_{h(t)+3}}(M_{\sigma\sigma'}). \end{aligned}$$

Similarly, the case $k = s$ can also be easily proved. Thus the proof is complete.

THEOREM 4.2. *If $C_{\sigma\sigma'} = \emptyset$ and $\widehat{C}_{\sigma\sigma'} = \emptyset$, then $M_{\sigma\sigma'}$ is the unique minimizing matrix on $\mathcal{F}(U_{\sigma\sigma'})$.*

Proof. First, we prove the theorem for $k_p \neq k'_p$. Note that if $n \leq 3$ then $C_{\sigma\sigma'} \neq \emptyset$ or $\widehat{C}_{\sigma\sigma'} \neq \emptyset$. Hence we may assume that $n \geq 4$. Let A be a minimizing matrix on $\mathcal{F}(U_{\sigma\sigma'})$. Since $U_{\sigma\sigma'}$ has a block representation (2.4), we have $A = M_{\sigma\sigma'}$ by averaging method of Lemma 4.2.

Now, we suppose that $C_{\sigma\sigma'} = \emptyset$ and $\widehat{C}_{\sigma\sigma'} = \emptyset$, and let B be an another minimizing matrix on $\mathcal{F}(U_{\sigma\sigma'})$, and we will show that $B = M_{\sigma\sigma'}$. Let

$$\begin{aligned} \bar{B} = (&I_1 \oplus J_{k_1-1} \oplus J_{k'_1-k_1-1} \oplus \cdots \oplus I_1 \oplus J_{k_p-k'_{p-1}-1} \oplus I_1 \oplus J_{k'_p-k_p-1})B \\ &(I_1 \oplus J_{j_1-2} \oplus I_1 \oplus J_{l_1-j_1} \oplus \cdots \oplus I_1 \oplus J_{j_p-l_{p-1}-2} \oplus I_1 \oplus J_{l_p-j_p}). \end{aligned}$$

Then $\bar{B}$ is also a minimizing matrix on $\mathcal{F}(U_{\sigma\sigma'})$ by Lemma 4.2 applied to the rows $2, \cdots, k_1,\ k_1+2, \cdots, k'_1, \cdots,\ k'_{p-1}+2, \cdots, k_p,\ k_p+2, \cdots, k'_p$ and columns $2, \cdots, j_1-1,\ j_1+1, \cdots, l_1, \cdots, l_{p-1}+2, \cdots, j_p-1,\ j_p+1, \cdots, l_p$.

Note that $I_1 \oplus J_{k'_t-k_t-1}$ for t such that $k_t = k'_t$ and $I_1 \oplus J_{j_s-l_{s-1}-2}$ for s such that $j_s = l_{s-1}+1$ do not appear.

Thus, we may assume, without loss of generality, that $B = \bar{B}$. Applying Lemma 4.1 to $\bar{B}$, then it is easily seen that $\bar{B} = M_{\sigma\sigma'}$.

It remains the case $k_p = k'_p$, which can be handled similarly. Hence the proof is completed.

THEOREM 4.3. *If $C_{\sigma\sigma'} \neq \emptyset$ or $\widehat{C}_{\sigma\sigma'} \neq \emptyset$, then $M^*_{\sigma\sigma'}$ is a minimizing matrix on $\mathcal{F}(U_{\sigma\sigma'})$.*

Proof. Since the submatrices S_i and S_j in $M^*_{\sigma\sigma'}$ for all $i,\ j \in \mathcal{I}$ do not have common rows and columns, it suffices to show that $M^*_{\sigma\sigma'}$ determined by only one submatrix S_i is a minimizing matrix on $\mathcal{F}(U_{\sigma\sigma'})$.

Let i be an odd number in $\mathcal{I}$. Then for $t \in C_{\sigma\sigma'}$ with $2t-1=i$, $t=1,\cdots,p$, we have

$$\begin{aligned} &M^*_{\sigma\sigma'}[1,\cdots,(\pi_{4t-4},\cdots,\pi_{4t-2})\hat{},\cdots,n|\cdot] \\ &= M_{\sigma\sigma'}[1,\cdots,(\pi_{4t-4},\cdots,\pi_{4t-2})\hat{},\cdots,n|\cdot] \end{aligned}$$

and

$$\begin{aligned} &M^*_{\sigma\sigma'}[\cdot|1,\cdots,(\pi_{4t-2},\cdots,\pi_{4t})\hat{},\cdots,n] \\ &= M_{\sigma\sigma'}[\cdot|1,\cdots,(\pi_{4t-2},\cdots,\pi_{4t})\hat{},\cdots,n] \end{aligned}$$

where $\hat{}$ stands for the deletion under it.
Thus, by Lemma 4.4 and (3.11), we get

$$\begin{aligned} \operatorname{per} M^*_{\sigma\sigma'} &= \left\{\prod_{i=1}^{4t-5}\Delta(i)\right\} P_{\hat{\pi}_{4t-5}}(M^*_{\sigma\sigma'}) \\ &= \left\{\prod_{i=1}^{4t-2}\Delta(i)\right\} P_{\hat{\pi}_{4t-2}}(M_{\sigma\sigma'}) \\ &= \prod_{i=1}^{p}\Delta(i) \\ &= \phi_{\sigma\sigma'}. \end{aligned}$$

Also, we can easily seen with similar argument for even number i in $\mathcal{I}$. Hence the proof is complete.

Theorem 4.3 tells us that, if $C_{\sigma\sigma'} \neq \emptyset$ or $\widehat{C}_{\sigma\sigma'} \neq \emptyset$, then there are infinitely many minimizing matrices on $\mathcal{F}(U_{\sigma\sigma'})$.

EXAMPLE 4.1. Let

$$\mathbf{y} = (14,\ 13,\ 13,\ 12,\ 11,\ 10,\ 10,\ 8,\ 8,\ 6,\ 6,\ 3,\ 3)^{\mathrm{T}}$$
$$\mathbf{x} = (16,\ 12,\ 12,\ 12,\ 12,\ 12,\ 9,\ 7,\ 7,\ 7,\ 7,\ 3,\ 1)^{\mathrm{T}}.$$

Then $\mathbf{y} \prec \mathbf{x}$ and we get the frames $\sigma = (3,\ 9,\ 13\ :\ 1,\ 7,\ 13)$ and $\sigma' = (4,\ 9,\ 13\ :\ 6,\ 11,\ 13)$. Note that $C_{\sigma\sigma'} = \emptyset$ and $\widehat{C}_{\sigma\sigma'} = \emptyset$. Thus

$$\begin{bmatrix}
1/3 & 2/15 & 2/15 & 2/15 & 2/15 & 2/15 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1/3 & 2/15 & 2/15 & 2/15 & 2/15 & 2/15 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1/3 & 2/15 & 2/15 & 2/15 & 2/15 & 2/15 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1/5 & 1/5 & 1/5 & 1/5 & 1/5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2/25 & 2/25 & 2/25 & 2/25 & 2/25 & 1/5 & 1/10 & 1/10 & 1/10 & 1/10 & 0 & 0 \\
0 & 2/25 & 2/25 & 2/25 & 2/25 & 2/25 & 1/5 & 1/10 & 1/10 & 1/10 & 1/10 & 0 & 0 \\
0 & 2/25 & 2/25 & 2/25 & 2/25 & 2/25 & 1/5 & 1/10 & 1/10 & 1/10 & 1/10 & 0 & 0 \\
0 & 2/25 & 2/25 & 2/25 & 2/25 & 2/25 & 1/5 & 1/10 & 1/10 & 1/10 & 1/10 & 0 & 0 \\
0 & 2/25 & 2/25 & 2/25 & 2/25 & 2/25 & 1/5 & 1/10 & 1/10 & 1/10 & 1/10 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/8 & 1/8 & 1/8 & 1/8 & 1/4 & 1/4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/8 & 1/8 & 1/8 & 1/8 & 1/4 & 1/4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/8 & 1/8 & 1/8 & 1/8 & 1/4 & 1/4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/8 & 1/8 & 1/8 & 1/8 & 1/4 & 1/4
\end{bmatrix}$$

is the unique minimizing matrix on $\mathcal{F}(U_{\sigma\sigma'})$. But, for

$$\mathbf{y} = (12,\ 12,\ 11,\ 10,\ 9,\ 8,\ 7,\ 6,\ 5,\ 5,\ 5,\ 3,\ 3)^{\mathrm{T}}$$
$$\mathbf{x} = (13,\ 12,\ 10,\ 10,\ 10,\ 8,\ 6,\ 6,\ 6,\ 6,\ 5,\ 2,\ 2)^{\mathrm{T}}$$

with $\mathbf{y} \prec \mathbf{x}$, we get $\sigma = (3,\ 7,\ 13\ :\ 2,\ 6,\ 13)$ and $\sigma' = (4,\ 8,\ 13\ :\ 5,\ 10,\ 13)$. Thus $C_{\sigma\sigma'} = \{1,\ 2\}$ and $\widehat{C}_{\sigma\sigma'} = \{1\}$. Therefore any doubly stochastic matrix of the form

$$\begin{bmatrix}
1/3 & 1/3 & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1/3 & 1/3 & & \ast & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1/3 & 1/3 & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1/3 & 1/3 & 1/3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1/9 & 1/9 & 1/9 & 1/3 & & & & & 0 & 0 & 0 \\
0 & 0 & 1/9 & 1/9 & 1/9 & 1/3 & & & \ast & & 0 & 0 & 0 \\
0 & 0 & 1/9 & 1/9 & 1/9 & 1/3 & & & & & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1/4 & 1/4 & 1/4 & 1/4 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1/10 & 1/10 & 1/10 & 1/10 & 1/5 & 1/5 & 1/5 \\
0 & 0 & 0 & 0 & 0 & 0 & 1/10 & 1/10 & 1/10 & 1/10 & 1/5 & 1/5 & 1/5 \\
0 & 0 & 0 & 0 & 0 & 0 & 1/10 & 1/10 & 1/10 & 1/10 & 1/5 & 1/5 & 1/5 \\
0 & 0 & 0 & 0 & 0 & 0 & 1/10 & 1/10 & 1/10 & 1/10 & 1/5 & 1/5 & 1/5 \\
0 & 0 & 0 & 0 & 0 & 0 & 1/10 & 1/10 & 1/10 & 1/10 & 1/5 & 1/5 & 1/5
\end{bmatrix}$$

or

$$\begin{bmatrix}
1/3 & 1/3 & 1/9 & 1/9 & 1/9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1/3 & 1/3 & 1/9 & 1/9 & 1/9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1/3 & 1/3 & 1/9 & 1/9 & 1/9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1/3 & 1/3 & 1/3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & & & & 1/3 & 1/12 & 1/12 & 1/12 & 1/12 & 0 & 0 & 0 \\
0 & 0 & & * & & 1/3 & 1/12 & 1/12 & 1/12 & 1/12 & 0 & 0 & 0 \\
0 & 0 & & & & 1/3 & 1/12 & 1/12 & 1/12 & 1/12 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1/4 & 1/4 & 1/4 & 1/4 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1/10 & 1/10 & 1/10 & 1/10 & 1/5 & 1/5 & 1/5 \\
0 & 0 & 0 & 0 & 0 & 0 & 1/10 & 1/10 & 1/10 & 1/10 & 1/5 & 1/5 & 1/5 \\
0 & 0 & 0 & 0 & 0 & 0 & 1/10 & 1/10 & 1/10 & 1/10 & 1/5 & 1/5 & 1/5 \\
0 & 0 & 0 & 0 & 0 & 0 & 1/10 & 1/10 & 1/10 & 1/10 & 1/5 & 1/5 & 1/5 \\
0 & 0 & 0 & 0 & 0 & 0 & 1/10 & 1/10 & 1/10 & 1/10 & 1/5 & 1/5 & 1/5
\end{bmatrix}$$

is a minimizing matrix on $\mathcal{F}(U_{\sigma\sigma'})$.

We close our discussion here by giving an answer to the problem of minimizing the permanent on some special class of majorization polytopes.

For $\mathbf{x} = (x_1, \cdots, x_n)^{\mathrm{T}}$ and $\mathbf{y} = (y_1, \cdots, y_n)^{\mathrm{T}}$ with $\mathbf{y} \prec \mathbf{x}$, let $\mathbf{y}^{(i)} = (y_{k'_{i-1}}, \cdots, y_{k_i})^{\mathrm{T}}$, $i = 1, \cdots, p$, where k_i and k'_i are defined in (2.1). If $z_1 = \cdots = z_n$ then a vector $\mathbf{z} = (z_1, \cdots, z_n)^{\mathrm{T}}$ is called a *scalar vector.*

THEOREM 4.5. *Let* $\mathbf{x} = (x_1, \cdots, x_n)^{\mathrm{T}}$ *and* $\mathbf{y} = (y_1, \cdots, y_n)^{\mathrm{T}}$ *in* $\mathbf{R}^n$. *Suppose that* $\mathbf{y} \prec \mathbf{x}$ *is not* k*-decomposable for each* $k = 1, \cdots, n-1$. *Then* $M_{\sigma\sigma'}$ *is a minimizing matrix over* $\Omega_n(\mathbf{y} \prec \mathbf{x})$ *iff* $\mathbf{y}^{(i)}$ *is the scalar vector for each* $i = 1, \cdots, p$. *Moreover, if* $C_{\sigma\sigma'} \neq \emptyset$ *or* $\widehat{C}_{\sigma\sigma'} \neq \emptyset$ *then* $M^*_{\sigma\sigma'}$ *holds above conclusion.*

Proof. Since $\Omega_n(\mathbf{y} \prec \mathbf{x}) \subset \mathcal{F}(U_{\sigma\sigma'})$ and $M_{\sigma\sigma'}$ is a minimizing matrix on $\mathcal{F}(U_{\sigma\sigma'})$, it is sufficient to show that $M_{\sigma\sigma'} \in \Omega_n(\mathbf{y} \prec \mathbf{x})$. It is readily proved from $\mathbf{y} = M_{\sigma\sigma'}\mathbf{x}$.

Also, we can easily seen with similar argument for $C_{\sigma\sigma'} \neq \emptyset$ or $\widehat{C}_{\sigma\sigma'} \neq \emptyset$.

PROBLEM A. Determine the permanental minimizing matrix on the majorization polytope $\Omega_n(\mathbf{y} \prec \mathbf{x})$.

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Department of Mathematics
Dae Jin University
Pocheon 487-800, Korea