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GENERALIZED FRACTIONS, GALOIS THEORY AND INJECTIVE ENVELOPES OF SIMPLE MODULES OVER POLYNOMIAL RINGS

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1. Introduction

In [9], we gave a very explicit description of the injective envelope of an arbitray simple module over a polynomial ring $K[X_1, \ldots, X_n]$ over a field K in indeterminates $X_1, \ldots, X_n$. This paper presents another approach to give a description.

Our another generalization is based on Galois theory, as well as modules of generalized fractions. Given a maximal ideal $\mathfrak{m}$ of the polynomial ring $K[X_1, \ldots, X_n]$, it is possible to find a finite, normal extension field L of K such that, if $\mathfrak{M}_1, \ldots, \mathfrak{M}_t$ denote the (necessarily finitely many) maximal ideals of $L[X_1, \ldots, X_n]$ whose contractions to $K[X_1, \ldots, X_n]$ are equal to $\mathfrak{m}$, then there exist $a_{ij} \in L$ $(i = 1, \ldots, t, j = 1, \ldots, n)$ such that

$$\mathfrak{M}_i = (X_1 - a_{i1}, \ldots, X_n - a_{in}) \quad \text{for all} \quad i = 1, \ldots, t.$$

We then use the results of [9, Section 2] to provide a description, in terms of modules of generalized fractions, of the injective envelope of the $L[X_1, \ldots, X_n]$-module $\oplus_{i=1}^{t} L[X_1, \ldots, X_n]/\mathfrak{M}_i$; it turns out that the Galois group G of L over K acts on this injective envelope in a natural way, and we shall show, in some cases, including the case where K has characteristic zero, that the 'fixed submodule' is naturally $K[X_1, \ldots, X_n]$-isomorphic to the injective envelope of the simple $K[X_1, \ldots, X_n]$-module $K[X_1, \ldots, X_n]/\mathfrak{m}$.

When discussing modules of generalized fractions, we shall use the same notation and terminology as in [9].

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2. Use of Galois theory

We begin by setting up notation which will be in force throughout the paper.

NOTATION AND TERMINOLOGY 2.1. Throughout the paper, we shall use K to denote a field, and A will denote $K[X_1, \ldots, X_n]$, the ring of polynomials over K in n indeterminates (where $n > 0$). Also, $\mathfrak{m}$ will denote a maximal ideal of A, and $\overline{K}$ will denote an algebraic closure of K.

If L is an algebraic extension field of K, then $A = K[X_1, \ldots, X_n]$ is a subring of $L[X_1, \ldots, X_n]$, and the latter ring is integral over A. We shall say that a prime ideal $\mathfrak{p}$ of $L[X_1, \ldots, X_n]$ *lies over* $\mathfrak{m}$ if $\mathfrak{p} \cap A = \mathfrak{m}$. Observe that such a $\mathfrak{p}$ must be maximal.

Let L be an algebraic extension field of K. We shall say that $\mathfrak{m}$ *splits in* L if the (necessarily finitely many (see 2.2 below)) maximal ideals $\mathfrak{M}_1, \ldots, \mathfrak{M}_t$ of $L[X_1, \ldots, X_n]$ which lie over $\mathfrak{m}$ are such that, for suitable $a_{ij} \in L$ $(i = 1, \ldots, t,\ j = 1, \ldots, n)$,

$$\mathfrak{M}_i = (X_1 - a_{i1}, \ldots, X_n - a_{in}) \quad \text{for all} \quad i = 1, \ldots, t.$$

LEMMA 2.2. *Let L be an algebraic extension field of K. Then there are only finitely many maximal ideals of $L[X_1, \ldots, X_n]$ which lie over* $\mathfrak{m}$.

Proof. The natural ring homomorphism $f : K[X_1, \ldots, X_n] \to L[X_1, \ldots, X_n]$ is faithfully flat, an the fibre ring of f over $\mathfrak{m}$ is isomorphic to $L \otimes_K A/\mathfrak{m}$: see [3, Section 2]. Note that this fibre ring is Noetherian, and by [4, 3.1], even Artinian. Hence, by [1, (2.2)], there are only finitely many maximal ideals of $L[X_1, \ldots, X_n]$ which lie over $\mathfrak{m}$.

LEMMA 2.3. *Let L be an algebraic extension field of K such that $\mathfrak{m}$ splits in L; let L' be an algebraic extension field of L. Then $\mathfrak{m}$ splits in L'.*

Proof. Let $\mathfrak{N}$ be a maximal ideal of $L'[X_1, \ldots, X_n]$ which lies over $\mathfrak{m}$. Then $\mathfrak{M} := \mathfrak{N} \cap L[X_1, \ldots, X_n]$ is a maximal ideal of $L[X_1, \ldots, X_n]$ which lies over $\mathfrak{m}$, and so there exist $a_1, \ldots, a_n \in L$ such that

$$\mathfrak{M} = \sum_{i=1}^{n} (X_i - a_i) L[X_1, \ldots, X_n].$$

But then $\mathfrak{N} \supseteq \sum_{i=1}^{n}(X_i - a_i)L'[X_1, \ldots, X_n]$, and since the latter ideal of $L'[X_1, \ldots, X_n]$ is maximal, it follows that

$$\mathfrak{N} = \sum_{i=1}^{n}(X_i - a_i)L'[X_1, \ldots, X_n].$$

LEMMA 2.4. *Suppose that L is an algebraic extension field of K and that $\mathfrak{m}$ splits in L. Let $\mathfrak{M}_1, \ldots, \mathfrak{M}_t$ be the maximal ideals of $L[X_1, \ldots, X_n]$ which lie over $\mathfrak{m}$, and suppose that $a_{ij} \in L$ ($i = 1, \ldots, t$, $j = 1, \ldots, n$) are such that*

$$\mathfrak{M}_i = (X_1 - a_{i1}, \ldots, X_n - a_{in}) \quad \textit{for all} \quad i = 1, \ldots, t.$$

Let K' be a subfield of L which contains $K(\{a_{ij} | i = 1, \ldots, t,\ j = 1, \ldots, n\})$. Then $\mathfrak{m}$ splits in K'.

Proof. By the 'Lying-over Theorem', each maximal ideal of $K'[X_1, \ldots, X_n]$ which lies over $\mathfrak{m}$ is the contraction of one of $\mathfrak{M}_1, \ldots, \mathfrak{M}_t$. Also, for each $i = 1, \ldots, t$.

$$\mathfrak{M}_i \cap K'[X_1, \ldots, X_n] = \sum_{j=1}^{n}(X_j - a_{ij})K'[X_1, \ldots, X_n].$$

LEMMA 2.5. *There exists a finite extension field L of K, with $K \subseteq L \subseteq \overline{K}$, such that $\mathfrak{m}$ splits in L.*

Proof. By 2.2, there are only finitely many maximal ideals of $\overline{K}[X_1, \ldots, X_n]$ which lie over $\mathfrak{m}$: let these be $\mathfrak{N}_1, \ldots, \mathfrak{N}_t$. By the Nullstellensatz, there exist $a_{ij} \in \overline{K}$ ($i = 1, \ldots, t$, $j = 1, \ldots, n$) such that

$$\mathfrak{N}_i = (X_1 - a_{i1}, \ldots, X_n - a_{in}) \quad \text{for all } i = 1, \ldots, t.$$

Then $K(a_{11}, \ldots, a_{1n}, a_{21}, \ldots, a_{tn})$ is, by 2.4, a suitable candidate for L.

REMARK 2.6. It follows from 2.5 and 2.3 that we can find a finite normal extension field L' of K, with $K \subseteq L' \subseteq \overline{K}$, such that $\mathfrak{m}$ splits in L'.

However, we cannot hope, in general, that we can always find such an extension of K which is also separable, as consideration of the following example shows. Let $\mathbb{F}_p(\tau)$ be a simple extension field of the field $\mathbb{F}_p$ of p elements, where p is a prime number, such that τ is transcendental over $\mathbb{F}_p$. Let $\mathfrak{m}_1$ be the maximal ideal $(X^p - \tau)$ of the polynomial ring $\mathbb{F}_p(\tau)[X]$. Let L be an algebraic extension field of $\mathbb{F}_p(\tau)$ in which $\mathfrak{m}_1$ splits, and let $b \in L$ be such that $(X - b)L[X]$ is a maximal ideal of $L[X]$ which lies over $\mathfrak{m}_1$. Now we can use 2.3 and enlarge L if necessary to be sure that it contains a p-th root $\tau^{1/p}$ of τ. Then

$$(X - \tau^{1/p})^p = X^p - \tau \in (X - b)L[X],$$

so that $X - \tau^{1/p} \in (X - b)L[X]$ and $\tau^{1/p} = b$. Thus L cannot be a separable extension of $\mathbb{F}_p(\tau)$.

ADDITIONAL NOTATION 2.7. For the remainder of this paper, we shall suppose that L is a finite extension field of K such that $\mathfrak{m}$ splits in L. We shall let $\mathfrak{M}_1, \ldots, \mathfrak{M}_t$ be the maximal ideals of $L[X_1, \ldots, X_n]$ which lie over $\mathfrak{m}$, and we shall suppose that $a_{ij} \in L$ ($i = 1, \ldots, t$, $j = 1, \ldots, n$) are such that

$$\mathfrak{M}_i = (X_1 - a_{i1}, \ldots, X_n - a_{in}) \quad \text{for all} \quad i = 1, \ldots, t.$$

We shall denote $L[X_1, \ldots, X_n]$ by B, and we shall let $G := Gal(L : K)$ denote the Galois group of L over K. Note that each $\sigma \in G$ induces an isomorphism of the ring B which has restriction to A equal to the identity and restriction to L equal to σ : we shall denote this induced isomorphism also by σ.

For each $i = 1, \ldots, t$, let

$$U_i := \{((X_1 - a_{i1})^{r_1}, \ldots, (X_n - a_{in})^{r_n}, 1) \mid r_i \in \mathbb{N} \quad \text{for all } i = 1, \ldots, n\},$$

and set $E'(B/\mathfrak{M}_i) = U_i^{-n-1}B$; by [9, 2.4], $E'(B/\mathfrak{M}_i)$ is an injective envelope of the B-module $B/\mathfrak{M}_i$. Lastly, set

$$E' := \oplus_{i=1}^{t} E'(B/\mathfrak{M}_i),$$

an injective B-module. Note that the comments in [9, 2.4] enable us to describe elements of E' in a very explicit manner.

REMARK 2.8. Let the situation be as in 2.1 and 2.7. Let $\sigma \in G$, and let $j \in \mathbb{N}$ with $1 \leq j \leq t$. Then $\sigma(\mathfrak{M}_j)$ must also be a maximal ideal of B which lies over $\mathfrak{M}$, and so must be $\mathfrak{M}_k$ for some k with $1 \leq k \leq t$. It follows that $\sigma(a_{ji}) = a_{ki}$ for all $i = 1, \ldots, n$.

It is easy to deduce from [7, 3.3(ii)] and [6, 2.2] that σ induces an isomorphism of A-modules $\sigma^{(j)} : E'(B/\mathfrak{M}_j) \to E'(B/\mathfrak{M}_j)$ which is such that

$$\sigma^{(j)}\left(\frac{h}{((X_1 - a_{j1})^{r_1}, \ldots, (X_n - a_{jn})^{r_n}, 1)}\right)$$
$$= \frac{\sigma(h)}{((X_1 - a_{k1})^{r_1}, \ldots, (X_n - a_{kn})^{r_n}, 1)}$$

for all $h \in B$, $r_1, \ldots, r_n \in \mathbb{N}$. It follows that σ induces an A-module automorphism of $E' = \oplus_{i=1}^{t} E'(B/\mathfrak{M}_i)$ which has, for all $j = 1, \ldots, t$, restriction to $E'(B/\mathfrak{M}_j)$ equal to $\sigma^{(j)}$. We shall denote this induced automorphism also by σ. Our plan is to study the 'fixed submodule'

$$E'^G := \{e' \in E' | \sigma(e') = e' \text{ for all } \sigma \in G\},$$

and show that, in certain circumstances, this A-module is an injective envelope of the simple A-module $A/\mathfrak{m}$.

REMARK 2.9. Let the situation be as in 2.1 and 2.7. It follows from [2, (3.5)] that E', when regarded as an A-module by restriction of scalars, is injective.

PROPOSITION 2.10. *Let the situation be as in 2.1 and 2.7. Assume in addition that $|G|$, the order of the Galois group G of L over K, is not divisible by char K, the characteristic of K. (This condition is of course satisfied if char $K = 0$.) Then*

$$E'^G = \{e' \in E' | \sigma(e') = e' \text{ for all } \sigma \in G\}$$

is an injective A-module.

Proof. The map $\theta : E' \to E'^G$ defined by

$$\theta(e') = \frac{1}{|G|} \sum_{\sigma \in G} \sigma(e') \quad \text{for all} \quad e' \in E'$$

is an A-homomorphism which satisfies $\theta \circ \imath = Id_{E'^G}$, where $\imath : E'^G \to E'$ denotes the inclusion map and $Id_{E'^G}$ denotes the identity map on E'^G. Hence E'^G is a direct summand of E' when the latter is considered as an A-module, and so, in view of 2.9, E'^G is an injective A-module.

3. The results

THEOREM 3.1. *Let the situation be as in 2.1 and 2.7. Assume that L is, in addition, a separable, normal extension of K.*

The Galois group G acts transitively on $\{\mathfrak{M}_1, \ldots, \mathfrak{M}_t\}$; for each $i = 1, \ldots, t$, let $\sigma_i \in G$ be such that $\sigma_i(\mathfrak{M}_1) = \mathfrak{M}_i$, with the understanding that σ_1 is the identity. Set $a_j = a_{1j}$ for all $j = 1, \ldots, n$, so that

$$\mathfrak{M}_1 = \sum_{i=1}^{n} (X_i - a_i)B$$

and

$$\mathfrak{M}_j = \sum_{i=1}^{n} (X_i - \sigma_j(a_i))B \quad \text{for all} \quad j = 2, \ldots, t.$$

Let $K' := K(a_1, \ldots, a_n)$. Observe that, with the notation of 2.7, U_1 is a triangular subset of $K'[X_1, \ldots, X_n]^{n+1}$, and, in view of [7, 3.3(ii)], [6, 2.2], and [1, 2.4], we can regard $U_1^{-n-1}K'[X_1, \ldots, X_n]$ as an A-submodule of $E'(B/\mathfrak{M}_1) = U_1^{-n-1}B$ in an obvious natural way. When this is done, with the notation of 2.8,

$$E'^G = \{(\delta, \sigma_2^{(1)}(\delta), \ldots, \sigma_t^{(1)}(\delta)) \in E' | \delta \in U_1^{-n-1}K'[X_1, \ldots, X_n]\}.$$

NOTE. By [9, 2.4], each element δ of $U_1^{-n-1}K'[X_1, \ldots, X_n]$ has a unique *dd*sum (with respect to $K'\backslash\{0\}$)

$$\delta = \sum_{i=1}^{w} \frac{k_i'}{((X_1 - a_1)^{\alpha_{i1}}, \ldots, (X_n - a_n)^{\alpha_{in}}, 1)},$$

where $w \in \mathbb{N}_0, k_1', \ldots, k_w' \in K'\backslash\{0\}$, and $(\alpha_{i1}, \ldots, \alpha_{in})$ $(i = 1, \ldots, w)$ are w distinct elements of $\mathbb{N}^n$. When this fact is combined with the result of 3.1 above, we obtain a very explicit description of a general element of E'^G.

Proof. The fact that G acts transitively on $\{\mathfrak{M}_1, \ldots, \mathfrak{M}_t\}$ is well known: see, for example, [5, Exercise 13.36].

First, let $e' = (\delta_1, \ldots, \delta_t) \in E'^G$, so that $\delta_1 \in E'(B/\mathfrak{M}_1) = U_1^{-n-1}B$: let

$$\delta_1 = \sum_{i=1}^{w} \frac{l_i}{((X_1 - a_1)^{\alpha_{i1}}, \ldots, (X_n - a_n)^{\alpha_{in}}, 1)},$$

where $w \in \mathbb{N}_0$, $l_1, \ldots, l_w \in L\backslash\{0\}$, and $(\alpha_{i1}, \ldots, \alpha_{in})$ $(i = 1, \ldots, w)$ are w distinct elements of $\mathbb{N}^n$, be the unique *dd*-sum for δ_1 with respect to $L\backslash\{0\}$. Our immediate aim is to show that $l_1, \ldots, l_w \in K'$ for all $i = 1, \ldots, w$.

Let $\sigma \in Gal(L : K')$, a subgroup of $G = Gal(L : K)$. Then $\sigma(\mathfrak{M}_1) = \mathfrak{M}_1$, and so, since $\sigma(e') = e'$, we must have $\sigma^{(1)}(\delta_1) = \delta_1$, that is

$$\sum_{i=1}^{w} \frac{\sigma(l_i)}{((X_1 - a_1)^{\alpha_{i1}}, \ldots, (X_n - a_n)^{\alpha_{in}}, 1)}$$
$$= \sum_{i=1}^{w} \frac{l_i}{((X_1 - a_1)^{\alpha_{i1}}, \ldots, (X_n - a_n)^{\alpha_{in}}, 1)}.$$

By the uniqueness of *dd*-sums with respect to $L\backslash\{0\}$ (see [9, 3.6]), it follows that $\sigma(l_i) = l_i$ for all $i = 1, \ldots, w$. As this is true for all $\sigma \in Gal(L : K')$, and as L is a finite, normal, separable extension of K, it follows from the Fundamental Theorem of Galois theory that $l_i \in K'$ for all $i = 1, \ldots, w$. Hence $\delta_1 \in U_1^{-n-1}K'[X_1, \ldots, X_n]$, and, since we must have, for each $i = 2, \ldots, t$, that $\delta_i = \sigma_i^{(1)}(\delta_1)$ simply because $e' = \sigma_i(e')$, we have proved that

$$E'^G \subseteq \{(\delta, \sigma_2^{(1)}(\delta), \ldots, \sigma_t^{(1)}(\delta)) \in E' | \delta \in U_1^{-n-1}K'[X_1, \ldots, X_n]\}.$$

Now let $e' = (\delta, \sigma_2^{(1)}(\delta), \ldots, \sigma_t^{(1)}(\delta))$, where $\delta \in U_1^{-n-1}K'[X_1, \ldots, X_n]$. We show that $e' \in E'^G$. Let $\sigma \in G$, and let $i \in \mathbb{N}$ with $1 \leq j \leq t$: we calculate the j-th component of $\sigma(e')$.

Let

$$\delta = \sum_{i=1}^{w} \frac{k_i'}{((X_1 - a_1)^{\alpha_{i1}}, \ldots, (X_n - a_n)^{\alpha_{in}}, 1)},$$

where $w \in \mathbb{N}_0$, $k'_1, \ldots, k'_w \in K'\backslash\{0\}$, and $(\alpha_{i1}, \ldots, \alpha_{in})(i = 1, \ldots, w)$ are w distinct elements of $\mathbb{N}^n$, be the unique *dd*-sum for δ with respect to $K'\backslash\{0\}$.

Let r be the unique integer between 1 and t for which $\sigma(\mathfrak{M}_r) = \mathfrak{M}_j$. Then the j-th component of $\sigma(e')$ is

$$\sigma^{(r)}(\sigma_r^{(1)}(\delta)) = \sum_{i=1}^{w} \frac{\sigma\sigma_r(k'_i)}{((X_1 - \sigma_j(a_1))^{\alpha_{i1}}, \ldots, (X_n - \sigma_j(a_n))^{\alpha_{in}}, 1)}.$$

But $\sigma_j^{-1}\sigma\sigma_r(\mathfrak{M}_1) = \mathfrak{M}_1$, and so

$$\sigma_j^{-1}\sigma\sigma_r(a_i) = a_i \quad \text{for all} \quad i = 1, \ldots, n.$$

Hence $\sigma_j^{-1}\sigma\sigma_r \in Gal(L : K')$ and so

$$\sigma\sigma_r(k'_i) = \sigma_j\sigma_j^{-1}\sigma\sigma_r(k'_i) = \sigma_j(k'_i) \quad \text{for all} \quad i = 1, \ldots, w.$$

Thus the j-th component of $\sigma(e')$ is

$$\sum_{i=1}^{w} \frac{\sigma_j(k'_i)}{((X_1 - \sigma_j(a_1))^{\alpha_{i1}}, \ldots, (X_n - \sigma_j(a_n))^{\alpha_{in}}, 1)} = \sigma_j^{(1)}(\delta),$$

the j-th component of e'. As this is true for all $j = 1, \ldots, t$, and also for all $\sigma \in G$, it follows that $e' \in E'^{G}$. Hence

$$E'^{G} \supseteq \{(\delta, \sigma_2^{(1)}(\delta), \ldots, \sigma_t^{(1)}(\delta)) \in E' | \delta \in U_1^{-n-1}K'[X_1, \ldots, X_n]\},$$

and the proof is complete.

THEOREM 3.2. *Let the situation be as in 2.1 and 2.7. Assume in addition that L is a separable, normal extension of K, and that $|G|$, the order of the Galois group G of L over K, is not divisible by char K, the characteristic of K. Then*

$$E'^{G} := \{e' \in E' | \sigma(e') = e' \quad \textit{for all} \quad \sigma \in G\}$$

is an injective envelope of the simple A-module $A/\mathfrak{m}$.

Proof. By 2.10, E'^G is an injective A-module. We shall use the description of E'^G obtained in 3.1 to show that it is an injective envelope of $A/\mathfrak{m}$. By 3.1, and with the notation thereof, the element

$$\zeta := \left(\frac{1}{((X_1 - a_1), \ldots, (X_n - a_n), 1)}, \frac{1}{((X_1 - \sigma_2(a_1)), \ldots, (X_n - \sigma_2(a_n)), 1)}, \ldots, \frac{1}{((X_1 - \sigma_t(a_1)), \ldots, (X_n - \sigma_t(a_n)), 1)}\right)$$

of E' actually belongs to E'^G; in view of [6, 2.2], it is not zero, and as it is annihilated by $\mathfrak{m}$ (by [7, 3.3(ii)]), it follows that $S := A\zeta$ is a simple A-submodule of E'^G and $S \cong A/\mathfrak{m}$. Our aim is to show that E'^G is an essential extension of S, as this will complete the proof.

With this in mind, let us now consider the effect on a generalized fraction

$$\frac{l}{((X_1 - a_1)^{\alpha_1}, \ldots, (X_n - a_n)^{\alpha_n}, 1)} \in E'(B/\mathfrak{M}_1) = U_1^{-n-1}B,$$

where $l \in L\backslash\{0\}$, $\alpha_1, \ldots, \alpha_n \in \mathbb{N}^n$ and $\alpha_1 > 1$, of multiplication by $(X_1 - c)$ for $c \in L$. First,

$$(X_1 - a_1)\frac{l}{((X_1 - a_1)^{\alpha_1}, \ldots, (X_n - a_n)^{\alpha_n}, 1)} = \frac{l}{((X_1 - a_1)^{\alpha_1 - 1}, (X_2 - a_2)^{\alpha_2}, \ldots, (X_n - a_n)^{\alpha_n}, 1)},$$

and this non-zero by [6, 2.2]. Secondly, for $c \in L$ with $c \neq a_1$,

$$(X_1 - c)\frac{l}{((X_1 - a_1)^{\alpha_1}, \ldots, (X_n - a_n)^{\alpha_n}, 1)}$$
$$= (X_1 - a_1 + a_1 - c)\frac{l}{((X_1 - a_1)^{\alpha_1}, \ldots, (X_n - a_n)^{\alpha_n}, 1)}$$
$$= \frac{l}{((X_1 - a_1)^{\alpha_1 - 1}, (X_2 - a_2)^{\alpha_2}, \ldots, (X_n - a_n)^{\alpha_n}, 1)} + \frac{(a_1 - c)l}{((X_1 - a_1)^{\alpha_1}, (X_2 - a_2)^{\alpha_2}, \ldots, (X_n - a_n)^{\alpha_n}, 1)},$$

and both terms in this last expression are non-zero (again by [6, 2.2]), so that this expression is actually a dd-sum with respect to $L\backslash\{0\}$.

Now let m_1 be the minimal polynomial of a_1 over K. Bearing in mind that L is a normal, separable extension of K, it follows from ideas like those in the preceding paragraph that

$$m_1(X_1)^{\alpha_1 - 1}\frac{l}{((X_1-a_1)^{\alpha_1},\ldots,(X_n-a_n)^{\alpha_n},1)}$$
$$=\frac{l'}{((X_1-a_1),(X_2-a_2)^{\alpha_2},\ldots,(X_n-a_n)^{\alpha_n},1)}$$

for some $l' \in L\backslash\{0\}$. Note also that

$$m_1(X_1)^{\alpha_1}\frac{l}{((X_1-a_1)^{\alpha_1},\ldots,(X_n-a_n)^{\alpha_n},1)}=0,$$

by [7, 3.3(ii)].

We now return to the problem of showing that E'^G is an essential extension of $S = A\zeta$. Let $e' \in E'^G$ with $e' \neq 0$, so that, by 3.1, and with the notation thereof, $e' = (\delta, \sigma_2^{(1)}(\delta), \ldots, \sigma_t^{(1)}(\delta))$ for some non-zero $\delta \in U_1^{-n-1}K'[X_1,\ldots,X_n]$. Bearing in mind [9, 2.4], let

$$\delta=\sum_{i=1}^{w}\frac{k'_i}{((X_1-a_1)^{\alpha_{i1}},\ldots,(X_n-a_n)^{\alpha_{in}},1)},$$

where $w \in \mathbb{N}$, $k'_1,\ldots,k'_w \in K'\backslash\{0\}$, and $(\alpha_{i1},\ldots,\alpha_{in})(i = 1,\ldots,w)$ are w distinct elements of $\mathbb{N}^n$, be the unique dd-sum for δ with respect to $K'\backslash\{0\}$. For each $i = 1,\ldots,n$, let m_i be the minimal polynomial of a_i over K. We can, and do, assume that the n-tuples $(\alpha_{i1},\ldots,\alpha_{in})(i = 1,\ldots,w)$ have been ordered so that, for each $i = 1,\ldots,w-1$, there exists $h_i \in \mathbb{N}$ with $1 \leq h_i \leq n$ such that $\alpha_{ij} = \alpha_{wj}$ for all $j = 1,\ldots,h_i - 1$ and $\alpha_i h_i < \alpha_{wh_i}$. Since $m_1(X_1)^{\alpha_{w1}-1}\ldots m_n(X_n)^{\alpha_{wn}-1} \in A$, it now follows from 3.1, [9, 3.6] and ideas like those in the preceding paragraph of this proof that

$$\begin{aligned} e'' &:= m_1(X_1)^{\alpha_{w1}-1} \dots m_n(X_n)^{\alpha_{wn}-1} e' \\ &= \Bigg(\frac{k'}{((X_1 - a_1), \dots, (X_n - a_n), 1)}, \\ &\qquad \frac{\sigma_2(k')}{((X_1 - \sigma_2(a_1)), \dots, (X_n - \sigma_2(a_n)), 1)}, \\ &\qquad \dots, \frac{\sigma_t(k')}{((X_1 - \sigma_t(a_1)), \dots, (X_n - \sigma_t(a_n)), 1)} \Bigg) \end{aligned}$$

for some $k' \in K' \backslash \{0\}$. But $K' = K(a_1, \dots, a_n)$ is a finite extension of K, and so there exists $f \in A$ such that $k'^{-1} = f(a_1, \dots, a_n)$. Also, the generalized fraction

$$\frac{k'}{((X_1 - a_1), \dots, (X_n - a_n), 1)}$$

is annihilated by $f - f(a_1, \dots, a_n)$. It follows from this that

$$\begin{aligned} fe'' = \Bigg(&\frac{k'^{-1}k'}{((X_1 - a_1), \dots, (X_n - a_n), 1)}, \\ &\frac{\sigma_2(k'^{-1}k')}{((X_1 - \sigma_2(a_1)), \dots, (X_n - \sigma_2(a_n)), 1)}, \\ &\dots, \frac{\sigma_t(k'^{-1}k')}{((X_1 - \sigma_t(a_1)), \dots, (X_n - \sigma_t(a_n)), 1)} \Bigg) = \zeta. \end{aligned}$$

Hence $Ae' \cap S \neq 0$, and so the proof is complete.

REMARK 3.3. We point out that, in the case in which K has characteristic 0, given a maximal ideal $\mathfrak{M}$ of $K[X_1, \dots, X_n]$, we can, by 2.6, find a finite normal extension field L of K, with $K \subseteq L \subseteq \overline{K}$, such that $\mathfrak{m}$ splits in L ; we can then use 3.2 to find a description for the injective envelope of the simple A-module $A/\mathfrak{m}$, and we can give precise descriptions of the elements of this injective envelope, in terms of *dd*-sums, by means of 3.1.

The reader might find it helpful if we compare the approaches given in [9, 4.3] and 3.3 in a fairly simple, but not completely trivial, example.

EXAMPLE 3.4. (cf. [9, 4.4]) Let $\mathfrak{m}$ be the maximal ideal $(X^2 - 2, Y^2 + 1)$ in the polynomial ring $\mathbb{Q}[X, Y] =: A$. By [9, 4.2 & 4.3],

$$E_{\mathbb{Q}[X,Y]}(\mathbb{Q}[X,Y]/\mathfrak{m}) \cong U^{-3}_{(X^2-2,Y^2+1,1)}A,$$

and each element of $U^{-3}_{(X^2-2,Y^2+1,1)}A$ can be written uniquely in the form

$$\sum_{j=1}^{\omega} \frac{a_j + b_jX + C_jY + d_jXY}{((X^2-2)^{\alpha_j}, (Y^2+1)^{\beta_j}, 1)},$$

where ω is a non-negative integer, (a_j, b_j, c_j, d_j) $(j = 1, \ldots, \omega)$ are ω elements of $\mathbb{Q}^4 \backslash \{(0,0,0,0)\}$ and (α_j, β_j) $(j = 1, \ldots, \omega)$ are ω distinct elements of $\mathbb{N}^2$.

On the other hand, an alternative description of $E_{\mathbb{Q}[X,Y]}(\mathbb{Q}[X,Y]/\mathfrak{m})$ is provided by 3.3, as follows. Let $L \in \mathbb{Q}(\sqrt{2}, i)$ and $B = L[X,Y]$. Let E' be the B-module

$$U^{-3}_{(X-\sqrt{2},Y-i,1)}B \oplus U^{-3}_{(X-\sqrt{2},Y+i,1)}B \oplus U^{-3}_{(X+\sqrt{2},Y-i,1)}B \oplus U^{-3}_{(X+\sqrt{2},Y+i,1}B$$

Then $E_{\mathbb{Q}[X,Y]}(\mathbb{Q}[X,Y]/\mathfrak{m})$ is isomorphic to the A-submodule of E' consisting of all elements which can be written (actually, in just one way) in the form

$$\Bigg(\sum_{j=1}^{\omega} \frac{a_j + b_j\sqrt{2} + c_ji + d_ji\sqrt{2}}{((X-\sqrt{2})^{\alpha_j}, (Y-i)^{\beta_j}, 1)}, \sum_{j=1}^{\omega} \frac{a_j + b_j\sqrt{2} - c_ji - d_ji\sqrt{2}}{((X-\sqrt{2})^{\alpha_j}, (Y+i)^{\beta_j}, 1)},$$
$$\sum_{j=1}^{\omega} \frac{a_j - b_j\sqrt{2} + c_ji - d_ji\sqrt{2}}{((X+\sqrt{2})^{\alpha_j}, (Y-i)^{\beta_j}, 1)}, \sum_{j=1}^{\omega} \frac{a_j - b_j\sqrt{2} - c_ji + d_ji\sqrt{2}}{((X+\sqrt{2})^{\alpha_j}, (Y+i)^{\beta_j}, 1)}\Bigg)$$

where ω is a non-integer, $(a_j, b_j, c_j, d_j)(j = 1, \ldots, \omega)$ are ω elements of $\mathbb{Q}^4 \backslash \{(0,0,0,0)\}$, and (α_j, β_j) $(j = 1, \ldots, \omega)$ are ω distinct elements of $\mathbb{N}^2$. It is intriguing that the bijection this submodule of E' and $U^{-3}_{(X^2-2,Y^2+1,1)}A$ whose existence is an obvious consequence of the above descriptions is not a $\mathbb{Q}[X,Y]$-isomorphism.

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