GENERALIZED FRACTIONS, GALOIS THEORY
AND INJECTIVE ENVELOPES OF SIMPLE
MODULES OVER POLYNOMIAL RINGS

YEONG MOO SONG

1. Introduction

In [9], we gave a very explicit description of the injective envelope of an arbitrary simple module over a polynomial ring \( K[X_1, \ldots, X_n] \) over a field \( K \) in indeterminates \( X_1, \ldots, X_n \). This paper presents another approach to give a description.

Our another generalization is based on Galois theory, as well as modules of generalized fractions. Given a maximal ideal \( m \) of the polynomial ring \( K[X_1, \ldots, X_n] \), it is possible to find a finite, normal extension field \( L \) of \( K \) such that, if \( \mathcal{M}_1, \ldots, \mathcal{M}_t \) denote the (necessarily finitely many) maximal ideals of \( L[X_1, \ldots, X_n] \) whose contractions to \( K[X_1, \ldots, X_n] \) are equal to \( m \), then there exist \( a_{ij} \in L \) \((i = 1, \ldots, t, j = 1, \ldots, n)\) such that

\[
\mathcal{M}_i = (X_1 - a_{i1}, \ldots, X_n - a_{in}) \quad \text{for all} \quad i = 1, \ldots, t.
\]

We then use the results of [9, Section 2] to provide a description, in terms of modules of generalized fractions, of the injective envelope of the \( L[X_1, \ldots, X_n] \)-module \( \bigoplus_{i=1}^t L[X_1, \ldots, X_n]/\mathcal{M}_i \); it turns out that the Galois group \( G \) of \( L \) over \( K \) acts on this injective envelope in a natural way, and we shall show, in some cases, including the case where \( K \) has characteristic zero, that the ‘fixed submodule’ is naturally \( K[X_1, \ldots, X_n] \)-isomorphic to the injective envelope of the simple \( K[X_1, \ldots, X_n] \)-module \( K[X_1, \ldots, X_n]/m \).

When discussing modules of generalized fractions, we shall use the same notation and terminology as in [9].

Received February 7, 1994.

1991 AMS Subject Classifications: 13B05, 13C11, 13E05.

Key words and phrases: modules of generalized fractions, injective module, dd-sum, Galois theory.
2. Use of Galois theory

We begin by setting up notation which will be in force throughout the paper.

**Notation and Terminology 2.1.** Throughout the paper, we shall use $K$ to denote a field, and $A$ will denote $K[X_1, \ldots, X_n]$, the ring of polynomials over $K$ in $n$ indeterminates (where $n > 0$). Also, $m$ will denote a maximal ideal of $A$, and $\overline{K}$ will denote an algebraic closure of $K$.

If $L$ is an algebraic extension field of $K$, then $A = K[X_1, \ldots, X_n]$ is a subring of $L[X_1, \ldots, X_n]$, and the latter ring is integral over $A$. We shall say that a prime ideal $p$ of $L[X_1, \ldots, X_n]$ lies over $m$ if $p \cap A = m$. Observe that such a $p$ must be maximal.

Let $L$ be an algebraic extension field of $K$. We shall say that $m$ splits in $L$ if the (necessarily finitely many (see 2.2 below)) maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_t$ of $L[X_1, \ldots, X_n]$ which lie over $m$ are such that, for suitable $a_{ij} \in L \ (i = 1, \ldots, t, \ j = 1, \ldots, n)$,

$$\mathfrak{m}_i = (X_1 - a_{i1}, \ldots, X_n - a_{in}) \quad \text{for all} \quad i = 1, \ldots, t.$$

**Lemma 2.2.** Let $L$ be an algebraic extension field of $K$. Then there are only finitely many maximal ideals of $L[X_1, \ldots, X_n]$ which lie over $m$.

*Proof.* The natural ring homomorphism $f : K[X_1, \ldots, X_n] \to L[X_1, \ldots, X_n]$ is faithfully flat, and the fibre ring of $f$ over $m$ is isomorphic to $L \otimes_K A/m$; see [3, Section 2]. Note that this fibre ring is Noetherian, and by [4, 3.1], even Artinian. Hence, by [1, (2.2)], there are only finitely many maximal ideals of $L[X_1, \ldots, X_n]$ which lie over $m$.

**Lemma 2.3.** Let $L$ be an algebraic extension field of $K$ such that $m$ splits in $L$; let $L'$ be an algebraic extension field of $L$. Then $m$ splits in $L'$.

*Proof.* Let $\mathfrak{M}$ be a maximal ideal of $L'[X_1, \ldots, X_n]$ which lies over $m$. Then $\mathfrak{M} := \mathfrak{M} \cap L[X_1, \ldots, X_n]$ is a maximal ideal of $L[X_1, \ldots, X_n]$ which lies over $m$, and so there exist $a_1, \ldots, a_n \in L$ such that

$$\mathfrak{M} = \sum_{i=1}^{n} (X_i - a_i)L[X_1, \ldots, X_n].$$
But then $\mathfrak{N} \supseteq \sum_{i=1}^{n} (X_i - a_i)L'[X_1, \ldots, X_n]$, and since the latter ideal of $L'[X_1, \ldots, X_n]$ is maximal, it follows that

$$\mathfrak{N} = \sum_{i=1}^{n} (X_i - a_i)L'[X_1, \ldots, X_n].$$

**Lemma 2.4.** Suppose that $L$ is an algebraic extension field of $K$ and that $\mathfrak{m}$ splits in $L$. Let $\mathfrak{M}_1, \ldots, \mathfrak{M}_t$ be the maximal ideals of $L[X_1, \ldots, X_n]$ which lie over $\mathfrak{m}$, and suppose that $a_{ij} \in L$ ($i = 1, \ldots, t$, $j = 1, \ldots, n$) are such that

$$\mathfrak{M}_i = (X_1 - a_{i1}, \ldots, X_n - a_{in}) \text{ for all } i = 1, \ldots, t.$$

Let $K'$ be a subfield of $L$ which contains $K(\{a_{ij} | i = 1, \ldots, t, \ j = 1, \ldots, n\})$. Then $\mathfrak{m}$ splits in $K'$.

**Proof.** By the 'Lying-over Theorem', each maximal ideal of $K'[X_1, \ldots, X_n]$ which lies over $\mathfrak{m}$ is the contraction of one of $\mathfrak{M}_1, \ldots, \mathfrak{M}_t$. Also, for each $i = 1, \ldots, t$.

$$\mathfrak{M}_i \cap K'[X_1, \ldots, X_n] = \sum_{j=1}^{n} (X_j - a_{ij})K'[X_1, \ldots, X_n].$$

**Lemma 2.5.** There exists a finite extension field $L$ of $K$, with $K \subseteq L \subseteq \overline{K}$, such that $\mathfrak{m}$ splits in $L$.

**Proof.** By 2.2, there are only finitely many maximal ideals of $\overline{K}[X_1, \ldots, X_n]$ which lie over $\mathfrak{m}$: let these be $\mathfrak{N}_1, \ldots, \mathfrak{N}_t$. By the Nullstellensatz, there exist $a_{ij} \in \overline{K}$ ($i = 1, \ldots, t$, $j = 1, \ldots, n$) such that

$$\mathfrak{N}_i = (X_1 - a_{i1}, \ldots, X_n - a_{in}) \text{ for all } i = 1, \ldots, t.$$

Then $K(a_{11}, \ldots, a_{1n}, a_{21}, \ldots, a_{tn})$ is, by 2.4, a suitable candidate for $L$. 
Remark 2.6. It follows from 2.5 and 2.3 that we can find a finite normal extension field $L'$ of $K$, with $K \subseteq L' \subseteq \overline{K}$, such that $m$ splits in $L'$.

However, we cannot hope, in general, that we can always find such an extension of $\overline{K}$ which is also separable, as consideration of the following example shows. Let $\mathbb{F}_p(\tau)$ be a simple extension field of the field $\mathbb{F}_p$ of $p$ elements, where $p$ is a prime number, such that $\tau$ is transcendental over $\mathbb{F}_p$. Let $m_1$ be the maximal ideal $(X^p - \tau)$ of the polynomial ring $\mathbb{F}_p(\tau)[X]$. Let $L$ be an algebraic extension field of $\mathbb{F}_p(\tau)$ in which $m_1$ splits, and let $b \in L$ be such that $(X - b)L[X]$ is a maximal ideal of $L[X]$ which lies over $m_1$. Now we can use 2.3 and enlarge $L$ if necessary to be sure that it contains a $p$-th root $\tau^{1/p}$ of $\tau$. Then

$$(X - \tau^{1/p})^p = X^p - \tau \in (X - b)L[X],$$

so that $X - \tau^{1/p} \in (X - b)L[X]$ and $\tau^{1/p} = b$. Thus $L$ cannot be a separable extension of $\mathbb{F}_p(\tau)$.

Additional Notation 2.7. For the remainder of this paper, we shall suppose that $L$ is a finite extension field of $K$ such that $m$ splits in $L$. We shall let $M_1, \ldots, M_t$ be the maximal ideals of $L[X_1, \ldots, X_n]$ which lie over $m$, and we shall suppose that $a_{ij} \in L$ ($i = 1, \ldots, n$, $j = 1, \ldots, n$) are such that

$$M_i = (X_1 - a_{i1}, \ldots, X_n - a_{in}) \quad \text{for all} \quad i = 1, \ldots, t.$$

We shall denote $L[X_1, \ldots, X_n]$ by $B$, and we shall let $G := Gal(L : K)$ denote the Galois group of $L$ over $K$. Note that each $\sigma \in G$ induces an isomorphism of the ring $B$ which has restriction to $A$ equal to the identity and restriction to $L$ equal to $\sigma$ : we shall denote this induced isomorphism also by $\sigma$.

For each $i = 1, \ldots, t$, let

$$U_i := \{(X_1 - a_{i1})^{r_1}, \ldots, (X_n - a_{in})^{r_n}, 1) \mid r_i \in \mathbb{N} \quad \text{for all} \quad i = 1, \ldots, n\},$$

and set $E'(B/M_i) = U_i^{-1}B$; by [9, 2.4], $E'(B/M_i)$ is an injective envelope of the $B$-module $B/M_i$. Lastly, set

$$E' := \bigoplus_{i=1}^t E'(B/M_i),$$

an injective $B$-module. Note that the comments in [9, 2.4] enable us to describe elements of $E'$ in a very explicit manner.
Remark 2.8. Let the situation be as in 2.1 and 2.7. Let $\sigma \in G$, and let $j \in \mathbb{N}$ with $1 \leq j \leq t$. Then $\sigma(\mathcal{M}_j)$ must also be a maximal ideal of $B$ which lies over $\mathcal{M}_i$, and so must be $\mathcal{M}_k$ for some $k$ with $1 \leq k \leq t$. It follows that $\sigma(a_{ji}) = a_{ki}$ for all $i = 1, \ldots, n$.

It is easy to deduce from [7, 3.3(ii)] and [6, 2.2] that $\sigma$ induces an isomorphism of $A$-modules $\sigma^{(j)} : E'(B/\mathcal{M}_j) \to E'(B/\mathcal{M}_j)$ which is such that

$$\sigma^{(j)} \left( \frac{h}{((X_1 - a_{j1})^{r_1}, \ldots, (X_n - a_{jn})^{r_n}, 1)} \right) = \frac{\sigma(h)}{((X_1 - a_{k1})^{r_1}, \ldots, (X_n - a_{kn})^{r_n}, 1)}$$

for all $h \in B$, $r_1, \ldots, r_n \in \mathbb{N}$. It follows that $\sigma$ induces an $A$-module automorphism of $E' = \bigoplus_{i=1}^{t} E'(B/\mathcal{M}_i)$ which has, for all $j = 1, \ldots, t$, restriction to $E'(B/\mathcal{M}_j)$ equal to $\sigma^{(j)}$. We shall denote this induced automorphism also by $\sigma$. Our plan is to study the ‘fixed submodule’

$$E'^G := \{ e' \in E' | \sigma(e') = e' \text{ for all } \sigma \in G \},$$

and show that, in certain circumstances, this $A$-module is an injective envelope of the simple $A$-module $A/m$.

Remark 2.9. Let the situation be as in 2.1 and 2.7. It follows from [2, (3.5)] that $E'$, when regarded as an $A$-module by restriction of scalars, is injective.

Proposition 2.10. Let the situation be as in 2.1 and 2.7. Assume in addition that $|G|$, the order of the Galois group $G$ of $L$ over $K$, is not divisible by $\text{char } K$, the characteristic of $K$. (This condition is of course satisfied if $\text{char } K = 0$.) Then

$$E'^G = \{ e' \in E' | \sigma(e') = e' \text{ for all } \sigma \in G \}$$

is an injective $A$-module.

Proof. The map $\theta : E' \to E'^G$ defined by

$$\theta(e') = \frac{1}{|G|} \sum_{\sigma \in G} \sigma(e') \text{ for all } e' \in E'$$
is an $A$-homomorphism which satisfies $\theta \circ i = Id_{E'G}$, where $i : E'G \to E'$ denotes the inclusion map and $Id_{E'G}$ denotes the identity map on $E'G$. Hence $E'G$ is a direct summand of $E'$ when the latter is considered as an $A$-module, and so, in view of 2.9, $E'G$ is an injective $A$-module.

3. The results

**Theorem 3.1.** Let the situation be as in 2.1 and 2.7. Assume that $L$ is, in addition, a separable, normal extension of $K$.

The Galois group $G$ acts transitively on $\{M_1, \ldots, M_t\}$: for each $i = 1, \ldots, t$, let $\sigma_i \in G$ be such that $\sigma_i(M_1) = M_i$, with the understanding that $\sigma_1$ is the identity. Set $a_j = a_{1j}$ for all $j = 1, \ldots, n$, so that

$$M_1 = \sum_{i=1}^{n} (X_i - a_i)B$$

and

$$M_j = \sum_{i=1}^{n} (X_i - \sigma_j(a_i))B \quad \text{for all} \quad j = 2, \ldots, t.$$ 

Let $K' := K(a_1, \ldots, a_n)$. Observe that, with the notation of 2.7, $U_1$ is a triangular subset of $K'[X_1, \ldots, X_n]^{n+1}$, and, in view of [7, 3.3(ii)], [6, 2.2], and [1, 2.4], we can regard $U_1^{-n-1}K'[X_1, \ldots, X_n]$ as an $A$-submodule of $E'(B/M_1) = U_1^{-n-1}B$ in an obvious natural way. When this is done, with the notation of 2.8,

$$E'G = \{(\delta, \sigma_2^{(1)}(\delta), \ldots, \sigma_t^{(1)}(\delta)) \in E' | \delta \in U_1^{-n-1}K'[X_1, \ldots, X_n]\}.$$ 

**Note.** By [9, 2.4], each element $\delta$ of $U_1^{-n-1}K'[X_1, \ldots, X_n]$ has a unique $dd$sum (with respect to $K'\setminus\{0\}$)

$$\delta = \sum_{i=1}^{w} \frac{k_i'}{((X_1 - a_1)^{\alpha_{i1}}, \ldots, (X_n - a_n)^{\alpha_{in}}, 1)},$$

where $w \in \mathbb{N}_0, k_1', \ldots, k_w' \in K'\setminus\{0\}$, and $(\alpha_{i1}, \ldots, \alpha_{in})$ ($i = 1, \ldots, w$) are $w$ distinct elements of $\mathbb{N}^n$. When this fact is combined with the
result of 3.1 above, we obtain a very explicit description of a general
element of $E^{G}$.

Proof. The fact that $G$ acts transitively on $\{\mathcal{M}_1, \ldots, \mathcal{M}_t\}$ is well
known: see, for example, [5, Exercise 13.36].

First, let $e' = (\delta_1, \ldots, \delta_t) \in E'^{G}$, so that $\delta_1 \in E'(B/\mathcal{M}_1) = U_1^{-n-1}B$.

Let

$$\delta_1 = \sum_{i=1}^{w} \frac{l_i}{((X_1 - a_1)^{\alpha_{i1}}, \ldots, (X_n - a_n)^{\alpha_{in}}, 1)},$$

where $w \in \mathbb{N}_0$, $l_1, \ldots, l_w \in L \setminus \{0\}$, and $(\alpha_{i1}, \ldots, \alpha_{in})$ ($i = 1, \ldots, w$) are

$w$ distinct elements of $\mathbb{N}^n$, be the unique $dd$-sum for $\delta_1$ with respect to $L \setminus \{0\}$. Our immediate aim is to show that $l_1, \ldots, l_w \in K'$ for all

$i = 1, \ldots, w$.

Let $\sigma \in Gal(L : K')$, a subgroup of $G = Gal(L : K)$. Then $\sigma(\mathcal{M}_1) = \mathcal{M}_1$, and so, since $\sigma(e') = e'$, we must have $\sigma^{(1)}(\delta_1) = \delta_1$, that is

$$\sum_{i=1}^{w} \frac{\sigma(l_i)}{((X_1 - a_1)^{\alpha_{i1}}, \ldots, (X_n - a_n)^{\alpha_{in}}, 1)}$$

$$= \sum_{i=1}^{w} \frac{l_i}{((X_1 - a_1)^{\alpha_{i1}}, \ldots, (X_n - a_n)^{\alpha_{in}}, 1)}.$$
where \( w \in \mathbb{N}_0, \ k'_1, \ldots, k'_w \in K' \setminus \{0\} \), and \((\alpha_{i1}, \ldots, \alpha_{in})(i = 1, \ldots, w)\) are \( w \) distinct elements of \( \mathbb{N}^n \), be the unique \( dd \)-sum for \( \delta \) with respect to \( K' \setminus \{0\} \).

Let \( r \) be the unique integer between 1 and \( t \) for which \( \sigma(\mathcal{M}_r) = \mathcal{M}_j \). Then the \( j \)-th component of \( \sigma(e') \) is

\[
\sigma^{(r)}(\sigma^{(1)}_r(\delta)) = \sum_{i=1}^w \frac{\sigma_r(k'_i)}{((X_1 - \sigma_j(a_1))^{\alpha_{i1}}, \ldots, (X_n - \sigma_j(a_n))^{\alpha_{in}}, 1)}.
\]

But \( \sigma_j^{-1}\sigma_r(\mathcal{M}_1) = \mathcal{M}_1 \), and so

\[
\sigma_j^{-1}\sigma_r(a_i) = a_i \quad \text{for all} \quad i = 1, \ldots, n.
\]

Hence \( \sigma_j^{-1}\sigma_r \in \text{Gal}(L : K') \) and so

\[
\sigma_r(k'_i) = \sigma_j\sigma_j^{-1}\sigma_r(k'_i) = \sigma_j(k'_i) \quad \text{for all} \quad i = 1, \ldots, w.
\]

Thus the \( j \)-th component of \( \sigma(e') \) is

\[
\sum_{i=1}^w \frac{\sigma_j(k'_i)}{((X_1 - \sigma_j(a_1))^{\alpha_{i1}}, \ldots, (X_n - \sigma_j(a_n))^{\alpha_{in}}, 1)} = \sigma^{(1)}_j(\delta),
\]

the \( j \)-th component of \( e' \). As this is true for all \( j = 1, \ldots, t \), and also for all \( \sigma \in G \), it follows that \( e' \in E'^{G} \). Hence

\[
E'^{G} \supseteq \{(\delta, \sigma_2^{(1)}(\delta), \ldots, \sigma_t^{(1)}(\delta)) \in E' | \delta \in U_1^{-n-1}K'[X_1, \ldots, X_n]\},
\]

and the proof is complete.

**Theorem 3.2.** Let the situation be as in 2.1 and 2.7. Assume in addition that \( L \) is a separable, normal extension of \( K \), and that \( |G| \), the order of the Galois group \( G \) of \( L \) over \( K \), is not divisible by \( \text{char} \ K \), the characteristic of \( K \). Then

\[
E'^{G} := \{e' \in E' | \sigma(e') = e' \quad \text{for all} \quad \sigma \in G\}
\]
is an injective envelope of the simple $A$-module $A/m$.

**Proof.** By 2.10, $E'^G$ is an injective $A$-module. We shall use the description of $E'^G$ obtained in 3.1 to show that it is an injective envelope of $A/m$. By 3.1, and with the notation thereof, the element

$$
\zeta := \frac{1}{((X_1 - a_1), \ldots, (X_n - a_n), 1)},
\frac{1}{((X_1 - \sigma_2(a_1)), \ldots, (X_n - \sigma_2(a_n)), 1)},
\ldots,
\frac{1}{((X_1 - \sigma_t(a_1)), \ldots, (X_n - \sigma_t(a_n)), 1)}
$$

of $E'$ actually belongs to $E'^G$; in view of [6, 2.2], it is not zero, and as it is annihilated by $m$ (by [7, 3.3(ii)]), it follows that $S := A\zeta$ is a simple $A$-submodule of $E'^G$ and $S \cong A/m$. Our aim is to show that $E'^G$ is an essential extension of $S$, as this will complete the proof.

With this in mind, let us now consider the effect on a generalized fraction

$$
l \frac{1}{((X_1 - a_1)^{\alpha_1}, \ldots, (X_n - a_n)^{\alpha_n}, 1)} \in E'(B/\mathfrak{M}_1) = U_1^{-n-1}B,
$$

where $l \in L \setminus \{0\}$, $\alpha_1, \ldots, \alpha_n \in \mathbb{N}$ and $\alpha_1 > 1$, of multiplication by $(X_1 - c)$ for $c \in L$. First,

$$
(X_1 - a_1)\frac{l}{((X_1 - a_1)^{\alpha_1}, \ldots, (X_n - a_n)^{\alpha_n}, 1)}
= \frac{l}{((X_1 - a_1)^{-\alpha_1-1}, (X_2 - a_2)^{\alpha_2}, \ldots, (X_n - a_n)^{\alpha_n}, 1)},
$$

and this non-zero by [6, 2.2]. Secondly, for $c \in L$ with $c \neq a_1$,

$$
(X_1 - c)\frac{l}{((X_1 - a_1)^{\alpha_1}, \ldots, (X_n - a_n)^{\alpha_n}, 1)}
= (X_1 - a_1 + a_1 - c)\frac{l}{((X_1 - a_1)^{\alpha_1}, \ldots, (X_n - a_n)^{\alpha_n}, 1)}
= \frac{l}{((X_1 - a_1)^{-\alpha_1-1}, (X_2 - a_2)^{\alpha_2}, \ldots, (X_n - a_n)^{\alpha_n}, 1)}
+ \frac{(a_1 - c)l}{((X_1 - a_1)^{\alpha_1}, (X_2 - a_2)^{\alpha_2}, \ldots, (X_n - a_n)^{\alpha_n}, 1)},
$$
and both terms in this last expression are non-zero (again by [6, 2.2]), so that this expression is actually a $dd$-sum with respect to $L \setminus \{0\}$.

Now let $m_1$ be the minimal polynomial of $a_1$ over $K$. Bearing in mind that $L$ is a normal, separable extension of $K$, it follows from ideas like those in the preceding paragraph that

$$m_1(X_1)^{\alpha_1 - 1} \frac{l}{((X_1 - a_1)^{\alpha_1}, \ldots, (X_n - a_n)^{\alpha_n}, 1)} = \frac{l'}{((X_1 - a_1), (X_2 - a_2)^{\alpha_2}, \ldots, (X_n - a_n)^{\alpha_n}, 1)}$$

for some $l' \in L \setminus \{0\}$. Note also that

$$m_1(X_1)^{\alpha_1} \frac{l}{((X_1 - a_1)^{\alpha_1}, \ldots, (X_n - a_n)^{\alpha_n}, 1)} = 0,$$

by [7, 3.3(ii)].

We now return to the problem of showing that $E'^G$ is an essential extension of $S = A\zeta$. Let $e' \in E'^G$ with $e' \neq 0$, so that, by 3.1, and with the notation thereof, $e' = (\delta, \sigma_2(1)(\delta), \ldots, \sigma_i(1)(\delta))$ for some non-zero $\delta \in U^{-n-1}_1 K'[X_1, \ldots, X_n]$. Bearing in mind [9, 2.4], let

$$\delta = \sum_{i=1}^w \frac{k_i'}{((X_1 - a_1)^{\alpha_{i1}}, \ldots, (X_n - a_n)^{\alpha_{in}}, 1)},$$

where $w \in \mathbb{N}$, $k'_1, \ldots, k'_w \in K' \setminus \{0\}$, and $(\alpha_{i1}, \ldots, \alpha_{in})(i = 1, \ldots, w)$ are $w$ distinct elements of $\mathbb{N}^n$, be the unique $dd$-sum for $\delta$ with respect to $K' \setminus \{0\}$. For each $i = 1, \ldots, n$, let $m_i$ be the minimal polynomial of $a_i$ over $K$. We can, and do, assume that the $n$-tuples $(\alpha_{i1}, \ldots, \alpha_{in})(i = 1, \ldots, w)$ have been ordered so that, for each $i = 1, \ldots, w - 1$, there exists $h_i \in \mathbb{N}$ with $1 \leq h_i \leq n$ such that $\alpha_{ij} = \alpha_{wj}$ for all $j = 1, \ldots, h_i - 1$ and $\alpha_i h_i < \alpha_{wh_i}$. Since $m_1(X_1)^{\alpha_{w1} - 1} \ldots m_n(X_n)^{\alpha_{wn} - 1} \in A$, it now follows from 3.1, [9, 3.6] and ideas like those in the preceding paragraph
of this proof that
\[ e'' := m_1(X_1)^{\alpha_{w_1}^{-1}} \ldots m_n(X_n)^{\alpha_{w_n}^{-1}} e' \]
\[ = \left( \frac{k'}{((X_1 - a_1), \ldots, (X_n - a_n), 1)}, \frac{\sigma_2(k')}{((X_1 - \sigma_2(a_1)), \ldots, (X_n - \sigma_2(a_n), 1)}, \ldots, \frac{\sigma_t(k')}{((X_1 - \sigma_t(a_1)), \ldots, (X_n - \sigma_t(a_n), 1)} \right) \]
for some \( k' \in K' \setminus \{0\} \). But \( K' = K(a_1, \ldots, a_n) \) is a finite extension of \( K \), and so there exists \( f \in A \) such that \( k'^{-1} = f(a_1, \ldots, a_n) \). Also, the generalized fraction
\[ k' \]
\[ ((X_1 - a_1), \ldots, (X_n - a_n), 1) \]
is annihilated by \( f - f(a_1, \ldots, a_n) \). It follows from this that
\[ fe'' = \left( \frac{k'^{-1} k'}{((X_1 - a_1), \ldots, (X_n - a_n), 1)}, \frac{\sigma_2(k'^{-1} k')}{((X_1 - \sigma_2(a_1)), \ldots, (X_n - \sigma_2(a_n), 1)}, \ldots, \frac{\sigma_t(k'^{-1} k')}{((X_1 - \sigma_t(a_1)), \ldots, (X_n - \sigma_t(a_n), 1)} \right) = \zeta. \]
Hence \( Ae' \cap S \neq 0 \), and so the proof is complete.

Remark 3.3. We point out that, in the case in which \( K \) has characteristic 0, given a maximal ideal \( \mathfrak{M} \) of \( K[X_1, \ldots, X_n] \), we can, by 2.6, find a finite normal extension field \( L \) of \( K \), with \( K \subseteq L \subseteq \overline{K} \), such that \( \mathfrak{m} \) splits in \( L \); we can then use 3.2 to find a description for the injective envelope of the simple \( A \)-module \( A/\mathfrak{m} \), and we can give precise descriptions of the elements of this injective envelope, in terms of \( dd \)-sums, by means of 3.1.

The reader might find it helpful if we compare the approaches given in [9, 4.3] and 3.3 in a fairly simple, but not completely trivial, example.
EXAMPLE 3.4. (cf. [9, 4.4]) Let \( m \) be the maximal ideal \( (X^2 - 2, Y^2 + 1) \) in the polynomial ring \( \mathbb{Q}[X, Y] =: A \). By [9, 4.2 & 4.3],

\[
E_{\mathbb{Q}[X,Y]}(\mathbb{Q}[X,Y]/m) \cong U_{(X^2 - 2, Y^2 + 1)}^{-3} A,
\]

and each element of \( U_{(X^2 - 2, Y^2 + 1)}^{-3} A \) can be written uniquely in the form

\[
\sum_{j=1}^{\omega} \frac{a_j + b_j X + c_j Y + d_j XY}{((X^2 - 2)^{\alpha_j}, (Y^2 + 1)^{\beta_j}, 1)},
\]

where \( \omega \) is a non-negative integer, \( (a_j, b_j, c_j, d_j) (j = 1, \ldots, \omega) \) are \( \omega \) elements of \( \mathbb{Q}^4 \setminus \{(0,0,0,0)\} \) and \( (\alpha_j, \beta_j) (j = 1, \ldots, \omega) \) are \( \omega \) distinct elements of \( \mathbb{N}^2 \).

On the other hand, an alternative description of \( E_{\mathbb{Q}[X,Y]}(\mathbb{Q}[X,Y]/m) \) is provided by 3.3, as follows. Let \( L \in \mathbb{Q}(\sqrt{2}, i) \) and \( B = L[X,Y] \). Let \( E' \) be the \( B \)-module

\[
U_{(X - \sqrt{2}, Y - i)}^{-3} B \oplus U_{(X - \sqrt{2}, Y + i)}^{-3} B \oplus U_{(X + \sqrt{2}, Y - i)}^{-3} B \oplus U_{(X + \sqrt{2}, Y + i)}^{-3} B
\]

Then \( E_{\mathbb{Q}[X,Y]}(\mathbb{Q}[X,Y]/m) \) is isomorphic to the \( A \)-submodule of \( E' \) consisting of all elements which can be written (actually, in just one way) in the form

\[
\left( \sum_{j=1}^{\omega} \frac{a_j + b_j \sqrt{2} + c_j i + d_j i \sqrt{2}}{((X - \sqrt{2})^{\alpha_j}, (Y - i)^{\beta_j}, 1)} \right), \sum_{j=1}^{\omega} \frac{a_j + b_j \sqrt{2} - c_j i - d_j i \sqrt{2}}{((X - \sqrt{2})^{\alpha_j}, (Y + i)^{\beta_j}, 1)}\right),
\]

\[
\sum_{j=1}^{\omega} \frac{a_j - b_j \sqrt{2} + c_j i - d_j i \sqrt{2}}{((X + \sqrt{2})^{\alpha_j}, (Y - i)^{\beta_j}, 1)}, \sum_{j=1}^{\omega} \frac{a_j - b_j \sqrt{2} - c_j i + d_j i \sqrt{2}}{((X + \sqrt{2})^{\alpha_j}, (Y + i)^{\beta_j}, 1)}\right)
\]

where \( \omega \) is a non-integer, \( (a_j, b_j, c_j, d_j)(j = 1, \ldots, \omega) \) are \( \omega \) elements of \( \mathbb{Q}^4 \setminus \{(0,0,0,0)\} \), and \( (\alpha_j, \beta_j) (j = 1, \ldots, \omega) \) are \( \omega \) distinct elements of \( \mathbb{N}^2 \). It is intriguing that the bijection this submodule of \( E' \) and \( U_{(X^2 - 2, Y^2 + 1)}^{-3} A \) whose existence is an obvious consequence of the above descriptions is not a \( \mathbb{Q}[X,Y] \)-isomorphism.

ACKNOWLEDGMENT. I am extremely grateful to professor R.Y. Sharp, the University of Sheffield, England, for his valuable advice and suggestions on this work.
References


Department of Mathematics Education
Sunchon National University
Sunchon, 540-742, Korea