J. Korean Math. Soc. **32** (1995), No. 2, pp. 279–287

LARGE DEVIATIONS FOR RANDOM WALKS WITH TIME STATIONARY RANDOM DISTRIBUTION FUNCTION

DUG HUN HONG

1. Introduction

Let $\mathcal{F}$ be a set of distributions on R with the topology of weak convergence, and let $\mathcal{A}$ be the σ-field generated by the open sets. We denote by $\mathcal{F}_1^\infty$ the space consisting of all infinite sequence $(F_1, F_2, \cdots)$, $F_n \in \mathcal{F}$ and R_1^∞ the space consisting of all infinite sequences $(x_1, x_2, \cdots)$ of real numbers. Take the σ–field $\mathcal{A}_1^\infty$ to be the smallest σ–field of subsets of $\mathcal{F}_1^\infty$ containing all finite–dimensional rectangles and take $\mathcal{B}_1^\infty$ to be the Borel σ–field of R_1^∞. Let $\omega = (F_1^\omega, F_2^\omega, \cdots)$ be the coordinate process in R_1^∞ and ν its distribution on $\mathcal{A}_1^\infty$. Let θ be the coordinate shift :$\theta^k(\omega) = \omega'$ with $F_n^{\omega'} = F_{n+k}^\omega$, $k = 1, 2, \cdots$. On $(R_1^\infty, \mathcal{B}_1^\infty)$ we also define the shift transformation $\sigma : R_1^\infty \to R_1^\infty$ by $\sigma(x_1, x_2, \cdots) = (x_2, x_3, \cdots)$. ν is called stationary if for every $A \in \mathcal{A}_1^\infty$, $\nu(\theta^{-1}(A)) = \nu(A)$ and we let π be its marginal distribution. Let $\mathcal{I}$ be the σ–field of invariant sets in $\mathcal{A}_1^\infty$, that is, $\mathcal{I} = \{A | \theta^{-1}(A) = A,\ A \in \mathcal{A}_1^\infty\}$ and let $\mathcal{J}$ be the σ–field of invariant sets in $\mathcal{B}_1^\infty$, that is, $\mathcal{J} = \{B | \sigma^{-1}(B) = B, B \in \mathcal{B}_1^\infty\}$. ν is called independent and identically distributed (i.i.d.) if ν is stationary and product measure. For each ω, define a probability measure P_ω on $(R_1^\infty, \mathcal{B}_1^\infty)$ so that $P_\omega = \Pi_{i=1}^\infty F_i^\omega$. A monotone class argument shows that $P_\omega(B), B \in \mathcal{B}_1^\infty$, is $\mathcal{A}_1^\infty$–measurable as a function of ω. So we can define a new probability measure such that $P(B) = \int P_\omega(B)\nu(d\omega)$. Define the process $\{X_n\}$ on $(R_1^\infty, \mathcal{B}_1^\infty)$ such that $X_n(x_1, x_2, \cdots) = x_n$ and set $S_n = X_1 + X_2 + \cdots + X_n$. By the definition of P_ω, $\{X_n\}$ are independent with respect to P_ω and we also note that $\{X_n\}$ is a sequence of independent and identically distributed random variables

---

Received February 23, 1994.
1991 AMS Subject Classifications: Primary 60F10, 60G20; Secondary 60G50.
Key words and phrases: Large deviation, random walks, stationarity.

sequence of independent and identically distributed random variables when $\mathcal{F}$ has just one element. In this paper we generalize the classical Cramer theorem in this set up.

2. Strong law of large numbers

In this section we consider some strong law of large numbers which are used to prove the main results.

LEMMA 1. *Let $\mathcal{F} = \{F \mid \int |x| dF(x) < \infty, \int x dF(x) = 0\}$, and let ν be stationary with $\int \int |x| dF(x) \pi(dF) < \infty$. Then X_1 with respect to P satisfies*

$$E[X_1 \mid \mathcal{J}] = 0 \qquad \text{a.s.}$$

Proof. By the assumption, $E|X_1| < \infty$ and hence $E[X_1 \mid \mathcal{J}]$ exists. Now let $A \in \mathcal{J}$ and let $\{(X_1, X_2, \cdots) \in B\} = A$ for some $B \in \mathcal{B}_1^\infty$. Then we have

$$\begin{aligned}
\int_A X_1 dP &= \int_{\{(X_1, X_2, \cdots) \in B\}} X_1 dP \\
&= \int_{\{(X_2, X_3, \cdots) \in B\}} X_1 dP \\
&= \int \left(\int x_1 dF_1^\omega(x_1) \int_B \Pi_{i=2}^\infty dF_i^\omega(x_i) \right) \nu(d\omega) \\
&= 0,
\end{aligned}$$

where the last equality holds because $\int x dF(x) = 0$ for all $F \in \mathcal{F}$. This proves the lemma.

THEOREM 1. *Let $\mathcal{F} = \{F \mid \int x dF(x) = 0, \int |x| dF(x) < \infty\}$ and ν be stationary with $\int \int |x| dF(x) \pi(dF) < \infty$. Then*

$$P_\omega\left\{\frac{S_n}{n} \to 0\right\} = 1, \quad \nu - \text{a.e. } \omega.$$

Proof. The proof follows directly from Proposition 1 and 3[5], Lemma 1, and the Birkhoff's ergodic theorem.

In general we then prove the following theorem.

THEOREM 2. *Let $\mathcal{F} = \{F | \int |x| dF(x) < \infty\}$ and let ν be stationary with $\int\int |x| dF(x)\pi(dF) < \infty$. Then*

$$P_\omega\left\{\frac{S_n}{n} \to E\left[\int x dF_1^{\cdot}(x)|\mathcal{I}\right](\omega)\right\} = 1, \qquad \nu-\text{a.e. } \omega.$$

($E[\int x dF_1^{\cdot}(x)|\mathcal{I}](\omega) = E[\int x dF_1^\omega(x)] = \int\int x dF(x)\pi(dF)$ in case ν is ergodic.)

Proof. By Theorem 1, $P_\omega\left\{\frac{S_n - E_\omega S_n}{n} \to 0\right\} = 1$, ν–a.e. ω, where $E_\omega S_n = \sum_{k=1}^n \int X_k dP_\omega = \sum_{k=1}^n \int x dF_k^\omega(x)$. We know $\dfrac{1}{n}E_\omega S_n \to E[\int x dF_1^{\cdot}(x)|\mathcal{I}](\omega)$, ν–a.e. ω by the ergodic theorem. Hence

$$P_\omega\left\{\frac{S_n}{n} \to E\left[\int x dF_1^{\cdot}(x)|\mathcal{I}\right](\omega)\right\} = 1, \qquad \nu-\text{a.e.}\omega.$$

3. Large deviations

We begin this section by introducing the logarithmic moment generating function $C_F(\xi) = \log M_F(\xi)$ where $M_F(\xi) = \int \exp(\xi x) dF(x)$, $\xi \in R$, and $C(\xi) = \int_{\mathcal{F}} C_F(\xi)\pi(dF)$, $\xi \in R$. Throughout this section we assume

$$M_F(\xi) < \infty \ \text{ for all} F \in \mathcal{F} \ \text{ and for all} \xi \in R, \tag{3.1}$$

$$C(\xi) < \infty \ \text{ for all} \xi \in R. \tag{3.2}$$

Note that since $\xi \in R \to C_F(\xi)$ is a convex function, for each $F \in \mathcal{F}$, so is $C(\xi)$. Next let $K(x)$ be the Legendre transform of $C(\xi)$:

$$K(x) \equiv \sup\{\xi x - C(\xi) | \xi \in R\}, \qquad x \in R. \tag{3.3}$$

Note that, by its definition as the pointwise supremum of linear functions, $K(x)$ is necessarily a convex function. In order to develop some feeling for the relationship between $C(\xi)$ and $K(x)$, we present the following elementary lemma.

LEMMA 2. *$K(x) \geq 0$, moreover, if $\int\int |x| dF(x)\pi(dF) < \infty$ and $p = \int\int x dF(x)\pi(dF)$ then $K(p) = 0$, K is non-decreasing on $[p, \infty)$ and non-increasing on $(-\infty, p]$. In addition, for $q \geq p$, $K(q) = \sup\{\xi q - C(\xi) | \xi \geq 0\}$ and for $q \leq p$, $K(q) = \sup\{\xi q - C(\xi) | \xi \leq 0\}$.*

Proof. We begin by noting that, since $\xi x - C(\xi) = 0$ for $\xi = 0$ and for every $x \in R$, $K(\xi) \geq 0$. Now suppose that $\int\int |x| dF(x)\pi(dF) < \infty$ and set $p = \int\int x dF(x)\pi(dF)$. To see that $K(p) = 0$, we use Jensen's inequality to obtain

$$\begin{aligned} C(\xi) &= \int_{\mathcal{F}} (\log \int \exp(\xi x) dF(x))\pi(dF) \\ &\geq \int_{\mathcal{F}} \int \xi x dF(x)\pi(dF) = \xi p \qquad \text{for all} \quad \xi \in R. \end{aligned}$$

In particular, this shows that $\xi p - C(\xi) \leq 0$ for all $\xi \in R$ and hence $K(p) \leq 0$. Since $K(x)$ is non-negative and convex, this proves that $K(p) = 0$, that $K(x)$ is non-decreasing on $[p, \infty)$, and that $K(x)$ is non-increasing on $(-\infty, p]$.

As a consequence of Lemma 2, we have the following.

LEMMA 3. *Let $\mathcal{F} = \{F | \int \exp(\xi x) dF(x) < \infty, \xi \in R\}$. If ν is stationary and ergodic with $\int\int |x| dF(x)\pi(dF) < \infty$, then for every closed set $G \subset R$,*

$$\limsup_{n\to\infty} \frac{1}{n} \log P_\omega\Big\{\frac{S_n}{n} \in G\Big\} \leq -\inf\{K(x) | x \in G\}, \qquad \nu - \text{a.e. } \omega.$$

Proof. Let $p = \int\int x dF(x)\pi(dF)$. Suppose $q \geq p (q \leq p)$. For $\xi \geq 0$,

$$\begin{aligned} P_\omega\Big\{\frac{S_n}{n} \geq q\Big\} &\leq \exp(-\xi q) E_\omega \exp\Big(\xi \frac{S_n}{n}\Big) \\ &= \exp(-\xi q) \Pi_{i=1}^n E_\omega \exp\Big(\xi \frac{X_i}{n}\Big). \end{aligned}$$

Then

$$\begin{aligned} &\frac{1}{n} \log P_\omega\Big\{\frac{S_n}{n} \geq q\Big\} \leq \frac{1}{n} \log(\exp(-\xi q) \Pi_{i=1}^n E_\omega\Big(\exp\Big(\xi \frac{X_i}{n}\Big)\Big) \\ &= -\frac{\xi}{n} q + \frac{1}{n} \sum_{i=1}^n \log \int \exp\Big(\frac{\xi x}{n}\Big) dF_i^\omega(x). \end{aligned}$$

Note that

$$\frac{1}{n}\sum_{i=1}^{n}\log\int\exp(\xi x)dF_i^{\omega}(x)\to\int(\log\int\exp(\xi x)dF(x))\pi(dF)=C(\xi)$$

ν–a.e. ω by the ergodic theorem. Then for given $\epsilon>0$, and ν–a.e. ω that we have for $n\geq N(\omega)$

$$\frac{1}{n}\log P_\omega\left\{\frac{S_n}{n}\geq q\right\}\leq\inf\{-\xi q+C(\xi)|\xi\geq 0\}+\epsilon.$$

Since ϵ is arbitrary, by Lemma 2,

$$\limsup_{n\to\infty}\frac{1}{n}\log P_\omega\left\{\frac{S_n}{n}\geq q\right\}\leq\inf\{-\xi q+C(q)|\xi\geq 0\}=-K(q).$$

Since K is non–decreasing(non–increasing) on $[p,\infty)$(on $(-\infty,p]$) above inequality proves the result when either $G\subset[p,\infty)$ or $G\subset(-\infty,p]$. On the other hand, if both $G\cap[p,\infty)\neq\phi$ and $G\cap(-\infty,p]\neq\phi$, let $q_+=\inf\{x\geq p|x\in G\}$ and $q_-=\sup\{x\leq p|x\in G\}$. Then for $\xi_1\geq 0,\ \xi_2\geq 0$

$$\begin{aligned}&P_\omega\left\{\frac{S_n}{n}\in G\right\}\\ \leq\ &\exp(-\xi_1 q_+)E_\omega\left(\exp\left(\xi_1\frac{S_n}{n}\right)\right)+\exp(\xi_2 q_-)E_\omega\left(\exp\left(-\xi_2\frac{S_n}{n}\right)\right)\end{aligned}$$

and hence

$$\begin{aligned}&\limsup_{n\to\infty}\frac{1}{n}\log P_\omega\left\{\frac{S_n}{n}\in G\right\}\\ \leq\ &\max\left[\limsup_{n\to\infty}\frac{1}{n}\log\left(\exp(-\xi_1 q_+)E_\omega(\exp(\xi_1\frac{S_n}{n}))\right),\right.\\ &\left.\limsup_{n\to\infty}\frac{1}{n}\log\left(\exp(\xi_2 q_-)E_\omega(\exp(-\xi_2\frac{S_n}{n}))\right)\right]\\ \leq\ &\max[-K(q_+),-K(q_-)]=-\inf\{K(x)|x\in G\}.\end{aligned}$$

For the lower bound we need the following lemma. We define $F^{-1}(t)=\sup\{x|F(x)\leq t\}$, $t\in(0,1)$.

LEMMA 4. *Suppose that for every $F \in \mathcal{F}$, F^{-1} is unbounded below and above and that there exists a measurable function $\phi(\xi, F)$ such that*

$$\left| \int \frac{x\exp(\xi x)}{M_F(\xi)} dF(x) \right| \le \phi(\xi, F), \tag{3.4}$$

$$\int \left(\sup_{|\xi| \le \xi_0} \phi(\xi, F) \right) \pi(dF) < \infty \qquad \textit{for all} \quad \xi_0 \in R. \tag{3.5}$$

Then we have

i) $f(\xi) = \int\int \frac{x\exp(\xi x)}{M_F(\xi)} dF(x)\pi(dF)$ *is continuous and* $\lim_{\xi\to\pm\infty} f(\xi) = \pm\infty$,

ii) *For each* q, $K(q) = \sup\{\xi q - c(\xi) | \xi \in R\}$ *is assumed at some point* $\xi = \xi_c(q)$ *or equivalently* $C'(\xi_c(q)) = q$.

Proof. i) We know that for all F, the function $\xi \to \int \frac{x\exp(\xi x)}{M_F(\xi)} dF(x)$ is continuous and $\int \frac{x\exp(\xi x)}{M_F(\xi)} dF(x) \to +\infty(-\infty)$ as $\xi \to +\infty(-\infty)$ by unboundedness of F^{-1}. So $f(\xi)$ is continuous by the Lebesgue dominated convergence theorem using (3.4) and (3.5) and we can easily check $f(\xi) \to +\infty(-\infty)$ as $\xi \to +\infty(-\infty)$.

ii) consider the following:

$$\begin{aligned}
&\lim_{\xi\to\xi'} \frac{C(\xi) - C(\xi')}{\xi - \xi'} \\
&= \lim_{\xi\to\xi'} \int \frac{C_F(\xi) - C_F(\xi')}{\xi - \xi'} \pi(dF) \\
&= \lim_{\xi\to\xi'} \int C_F'(\xi'')\pi(dF), \quad \text{where} \quad \xi'' \in (\xi, \xi') \quad \text{or} \quad \xi'' \in (\xi', \xi) \\
&= \lim_{\xi\to\xi'} \int\int \frac{x\exp(\xi'' x)}{M_F(\xi'')} dF(x)\pi(dF) \\
&= \int \lim_{\xi\to\xi'} \int \frac{x\exp(\xi'' x)}{M_F(\xi'')} dF(x)\pi(dF) \\
&= \int\int \frac{x\exp(\xi' x)}{M_F(\xi')} dF(x)\pi(dF) = f(\xi');
\end{aligned}$$

hence $C'(\xi) = f(\xi)$.

The fourth equality above follows from (3.4), (3.5) and the dominated convergence theorem. Note that C is convex. So for every given q there exists $\xi_c(q)$ such that $C'(\xi_c(q)) = q$. This proves the lemma.

THEOREM 3. *Suppose that ν is stationary and ergodic. Then under (3.1), (3.2), (3.4) and (3.5) for every measurable $\Gamma \subset R$ we have that*

$$\begin{aligned} &-\inf\{K(x)|x \in \Gamma^o\} \\ &\leq \liminf_{n\to\infty} \frac{1}{n} \log P_\omega\Big(\frac{S_n}{n} \in \Gamma\Big) \leq \limsup_{n\to\infty} \frac{1}{n} \log P_\omega\Big(\frac{S_n}{n} \in \Gamma\Big) \\ &\leq -\inf\{K(x)|x \in \overline{\Gamma}\} \qquad \nu - a.e.\ \omega. \end{aligned}$$

Proof. In view of Lemma 3, we only need to show that if $q \in R$ and $\delta > 0$,

$$\liminf_{n\to\infty} \frac{1}{n} \log P_\omega\Big(\frac{S_n}{n} \in (q-\epsilon, q+\epsilon)\Big) \geq -K(q). \tag{3.6}$$

In proving (3.6), we first suppose that for all $F \in \mathcal{F}$, F^{-1} is unbounded above and below. Then for each q, there exists ξ such that $C'(\xi) = f(\xi) = q$ by Lemma 4, and so $K(q) = \xi q - C(\xi)$.

$$\begin{aligned} &P_\omega\Big\{\frac{S_n}{n} \in (q-\delta, q+\delta)\Big\} \\ &= \int_{\left\{\frac{x_1+\cdots+x_m}{n} \in (q-\delta,q+\delta)\right\}} dF_1^\omega(x_1)\cdots dF_n^\omega(x_n) \\ &\geq M_{F_1^\omega}(\xi)\cdots M_{F_n^\omega}(\xi)\exp(-\xi(q+\delta)n) \\ &\quad\times \int_{\left\{\frac{x_1+\cdots+x_n}{n} \in (q-\delta,q+\delta)\right\}} \\ &\quad \frac{\exp(\xi x_1)}{M_{F_1^\omega}(\xi)} dF_1^\omega(x_1)\cdots \frac{\exp(\xi x_n)}{M_{F_n^\omega}(\xi)} dF_n^\omega(x_n). \end{aligned}$$

Here we need the following lemma.

LEMMA 5. *Under the conditions of Theorem 3, we have for ν–a.e. ω*

$$\int_{\left\{\frac{x_1+\cdots+x_n}{n}\in(q-\delta,q+\delta)\right\}} \frac{\exp(\xi x_1)}{M_{F_1^\omega}(\xi)}dF_1^\omega(x_1)\cdots\frac{\exp(\xi x_n)}{M_{F_n^\omega}(\xi)}dF_n^\omega(x_n)\to 1,$$
$$\text{as}\quad n\to\infty.$$

Proof. For given ξ define $\dot{F}$ so that $\dot{F}(t)=\int_{-\infty}^{t}\frac{\exp(\xi x)}{M_F(\xi)}dF(x)$. Let $\dot{\mathcal{F}}=\{\dot{F}|F\in\mathcal{F}\}$. Define $\phi:\mathcal{F}_1^\infty\to\dot{\mathcal{F}}_1^\infty$ by $\phi(\omega)=\dot{\omega}=(\dot{F}_1^\omega,\dot{F}_2^\omega,\cdots)$. Now let $\dot{\nu}=\nu\circ\phi^{-1}$. Then $\dot{\nu}$ is stationary and ergodic. Now we apply Theorem 2 to this probability measure. Then we have

$$P_{\dot{\omega}}\left\{\frac{S_n}{n}\to\int\int x d\dot{F}(x)\pi(dF)\right\}=1\qquad \nu-\text{a.e. }\omega,$$

Note that $\int\int x d\dot{F}(x)\pi(dF)=\int\int\frac{x\exp(\xi x)}{M_F(\xi)}dF(x)\pi(dF)=f(\xi)=q$. Hence the lemma follows.

Now back to the proof of Theorem 3. By the above lemma we have, for given $\epsilon>0$, and ν–a.e. ω that for $n\ge N(\omega)$

$$\frac{1}{n}\log P_\omega\left\{\frac{S_n}{n}\in(q-\delta,q+\delta)\right\}\ge\frac{1}{n}\sum_{i=1}^{n}\log M_{F_i^\omega}(\xi)-\xi(q+\delta)-\epsilon$$

and consequently

$$\liminf_{n\to\infty}\frac{1}{n}\log P_\omega\left\{\frac{S_n}{n}\in(q-\delta,q+\delta)\right\}\ge C(\xi)-\xi(q+\delta),\qquad \nu-\text{a.e. }\omega.$$

By monotonicity, the result holds with $\delta=0$, *i.e.*, with $K(q)$.

We must now handle the general case. Suppose that there exists $F\in\mathcal{F}$ such that F^{-1} is bounded. We replace all F by the distribution $F*\phi_\epsilon$ where $\phi_\epsilon(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x/\epsilon}\exp\left(-\frac{y^2}{2}\right)dy$ and apply the above results to this. Here $*$ is the convolution. Then, letting $\epsilon\downarrow 0$ the desired result follows.

REMARK 1. If ν is i.i.d., then $\{X_n\}$ is i.i.d. with respect to P with distribution function $\overline{F}_1(x) = \int F(x)\pi(dF)$. By the Cramer theorem, $\{X_n\}$ with respect to P has the rate function

$$\overline{K}(x) = \sup\{\xi c - \overline{C}(\xi) | \xi \in R\}.$$

where $\overline{C}(\xi) = \log \int \exp(\xi x) d\overline{F}(x) = \log \int \int \exp(\xi x) dF(x)\pi(dF)$. By Jensen's inequality, we can check easily $\overline{C}(\xi) \geq C(\xi)$ and hence $\overline{K}(x) \leq K(x)$.

References

1. L. Brieman, ***Probability***, Addison-Wesely, Reading, Mass, 1968.
2. K. L. Chung, ***A course in probability theory***, 2nd ed., Acadimic Press, New York London, 1974.
3. J. D. Dueschel and D. W. Strook, *Large deviations*, Academic Press, 1989.
4. D. H. Hong, ***Random walks with time stationary random distribution function***, Ph.D. Thesis, Univ. of Minnesota., 1990.
5. D. H. Hong and J. S. Kwon, *An LIL for random walks with time stationary random distribution function*, Yokohama Math. J. **40** (1993), 115-120.

Department of Statistics
Taegu Hyosung Catholic University
Kyungbuk 713-702, Korea